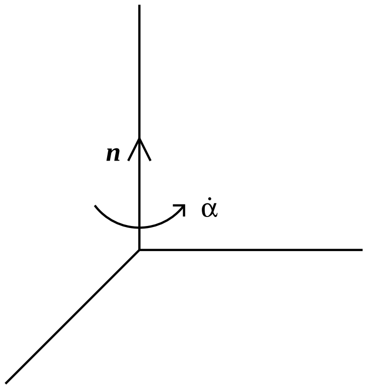
RIGID BODY ROTATION

THE MOMENT OF INERTIA TENSOR

A rigid body has six degrees of freedom, three giving the position of the centre of mass, and three specifying the orientation. The simplest form of rotation is rotation of a rigid body about a fixed axis with angular velocity $\boldsymbol{\omega}$. The vector angular velocity is defined as

$$\boldsymbol{\omega} = \boldsymbol{n}\ \dot{\alpha} \quad ,$$

where $\boldsymbol{n}$ is a unit vector specifying the direction of the axis of rotation and $\dot{\alpha}$ is the rate at which angle α is swept out around the axis in the sense in which a right hand screw would advance along the direction of $\boldsymbol{n}$.



The rigid body is composed of many point masses m_i each of which we assume to be located at position $\boldsymbol{r}_i$ with respect to a fixed origin O on the axis of rotation. The effect of rotation is that each particle sweeps out a circle around the axis of rotation and its position vector rigidly sweeps around the axis of rotation as well. The consequence is that the velocity of any particle m_i has the same simple form

$$\frac{d\boldsymbol{r}_i}{dt} = \boldsymbol{v}_i = \boldsymbol{\omega} \times \boldsymbol{r}_i \quad .$$

This special form for the velocity enables us to re-express the kinetic energy of a rigid body in a special way.

$$T = \sum_i \frac{1}{2} m_i \boldsymbol{v}_i^2 = \frac{1}{2}\sum_i m_i \boldsymbol{v}_i \cdot \boldsymbol{v}_i = \frac{1}{2}\sum_i m_i \boldsymbol{v}_i \cdot (\boldsymbol{\omega} \times \boldsymbol{r}_i) = \frac{1}{2}\sum_i \boldsymbol{\omega} \cdot (\boldsymbol{r}_i \times m_i \boldsymbol{v}_i) \quad ,$$

or

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot \left(\sum_i \boldsymbol{r}_i \times m_i \boldsymbol{v}_i\right) = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{L} \quad ,$$

where $\boldsymbol{L}$ is the total angular momentum of the rigid body with respect to the origin O on the axis of rotation from which positions are measured.

Next we can ask if the angular momentum $\boldsymbol{L}$ can be expressed in terms of the angular velocity $\boldsymbol{\omega}$. To analyze this we write

$$\boldsymbol{L} = \sum_i m_i (\boldsymbol{r}_i \times \boldsymbol{v}_i) = \sum_i m_i [\boldsymbol{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{r}_i)] \quad ,$$

or, using an identity for the triple vector product, $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$,

$$\boldsymbol{L} = \sum_i m_i[(\boldsymbol{r}_i \cdot \boldsymbol{r}_i)\boldsymbol{\omega} - (\boldsymbol{r}_i \cdot \boldsymbol{\omega})\boldsymbol{r}_i] \quad .$$

From this expression we see that $\boldsymbol{L}$ is a linear function of $\boldsymbol{\omega}$. To see this in more detail we can look at one component of $\boldsymbol{L}$, say the x - component and expand the scalar products above to get

$$L_x = \sum_i m_i[(x_i^2 + y_i^2 + z_i^2)\omega_x - (x_i\omega_x + y_i\omega_y + z_i\omega_z)x_i] \quad ,$$

$$L_x = \sum_i m_i[x_i^2\omega_x + y_i^2\omega_x + z_i^2\omega_x - x_i^2\omega_x - y_i\omega_y x_i - z_i\omega_z x_i] \quad ,$$

$$L_x = \left(\sum_i m_i(y_i^2 + z_i^2)\right)\omega_x + \left(-\sum_i m_i x_i y_i\right)\omega_y + \left(-\sum_i m_i x_i z_i\right)\omega_z \quad .$$

A similar argument goes through for the y and z components of $\boldsymbol{L}$ so that we can write the linear relationship between $\boldsymbol{L}$ and $\boldsymbol{\omega}$ as

$$\begin{aligned} L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \quad , \\ L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \quad , \\ L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \quad , \end{aligned}$$

where the coefficients I_{ij} are defined by

$$\begin{aligned} I_{xx} &= \sum_i m_i(y_i^2 + z_i^2) \quad , \quad I_{yy} = \sum_i m_i(x_i^2 + z_i^2) \quad , \quad I_{zz} = \sum_i m_i(x_i^2 + y_i^2) \\ I_{xy} &= -\sum_i m_i x_i y_i = I_{yx} \\ I_{xz} &= -\sum_i m_i x_i z_i = I_{zx} \\ I_{yz} &= -\sum_i m_i y_i z_i = I_{zy} \end{aligned} \quad .$$

This relation can be thought of in terms of matrix multiplication as

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \quad .$$

PRINCIPAL AXIS SYSTEM

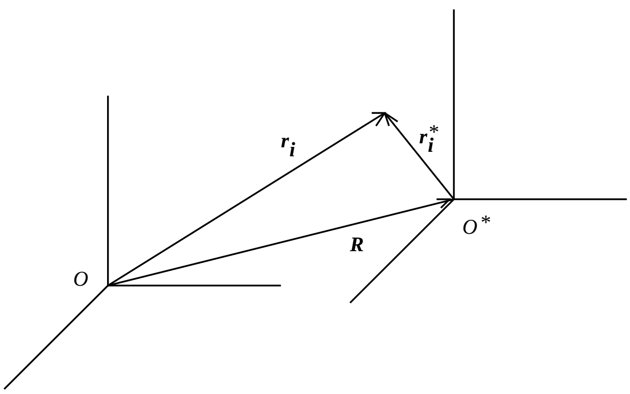
The important fact that this result demonstrates is that the angular momentum $\boldsymbol{L}$ in general does not point in the same direction as the angular velocity $\boldsymbol{\omega}$. The relation between them depends on the nine component object $\boldsymbol{I}$ represented above as a matrix. the quantity $\boldsymbol{I}$ is called the moment of inertia tensor and is an example of a symmetric (the matrix is symmetric) second rank (nine components) tensor. In the most general situation, all nine elements of $\boldsymbol{I}$ are non-vanishing. However, from the definitions of the elements I_{ij}, it is clear that the values of these components depend on the coordinate system we have chosen. In a different coordinate system the values would be different. Because $\boldsymbol{I}$ is symmetric, it is a mathematical theorem that there is always at least one coordinate system in which $\boldsymbol{I}$ is diagonal, i.e., I_{xx}, I_{yy}, I_{zz} are non-vanishing but I_{xy}, I_{xz}, I_{yz} are all zero. Such a coordinate system is called a *principal axis system* for the rigid body. In such a coordinate system, each coordinate axis is a principal axis. We can also characterise a principal axis by saying that it is a direction such that if we rotate the object about that direction, the angular momentum and angular velocity will be parallel.

There are symmetry reasons why some of the off-diagonal elements of $\boldsymbol{I}$ may vanish. For example, if the $x-y$ plane is a plane in which the rigid body is reflection symmetric, then for each mass m_i at position (x_i, y_i, z_i) there must be an identical mass m_i at the mirror position $(x_i, y_i, -z_i)$. It is easy to see from the definition of I_{xz} or I_{yz} that the terms must then cancel in pairs $(-m_i x_i z_i - m_i x_i(-z_i) = 0)$ so that $I_{xz} = 0$ and by the same argument $I_{yz} = 0$. If we can find two planes of mirror symmetry at right angles to each other, then all off-diagonal elements of $\boldsymbol{I}$ will vanish giving a principle axis system. If the object has an axis of rotational symmetry, it is easy to see that it must also have two mirror symmetry planes which meet at right angles and whose intersection is the axis of rotational symmetry thereby again giving a principal axis system.

PARALLEL AXIS THEOREM

If we choose a different coordinate system, the elements of $\boldsymbol{I}$ will be different. Sometimes we can use this to our advantage. It is usually simplest to work out the elements of $\boldsymbol{I}$ in the centre of mass frame where we denote it by $\boldsymbol{I}^*$. If we have an origin O elsewhere, we can relate $\boldsymbol{I}$ calculated at O to $\boldsymbol{I}^*$ calculated in a *parallel coordinate system* with origin at the centre of mass O^*. If $\boldsymbol{R}$ is the translation vector taking us from origin O to origin O^* then the position vectors of masses m_i in the two frames of reference are related by

$$\boldsymbol{r}_i = \boldsymbol{R} + \boldsymbol{r}_i^* \quad .$$



We then calculate

$$I_{xx} = \sum_i m_i(y_i^2 + z_i^2) = \sum_i m_i[(R_y + y_i^*)^2 + (R_z + z_i^*)^2] \quad ,$$

$$I_{xx} = \sum_i m_i[R_y^2 + R_z^2 + 2R_y y_i^* + 2R_z z_i^* + (y_i^*)^2 + (z_i^*)^2] \quad ,$$

but, because in the centre of mass frame we have by its definition that

$$\sum_i m_i \boldsymbol{r}_i^* = 0 \quad ,$$

the cross-terms above vanish,

$$\sum_i m_i 2R_y y_i^* = 2R_y \sum_i m_i y_i^* = 0 \quad , \quad \sum_i m_i 2R_z z_i^* = 2R_z \sum_i m_i z_i^* = 0 \quad ,$$

giving

$$I_{xx} = M(R_y^2 + R_z^2) + \sum_i m_i[(y_i^*)^2 + (z_i^*)^2] \quad ,$$

or

$$I_{xx} = M(R_y^2 + R_z^2) + I_{xx}^* \quad .$$

Similarly, for an off diagonal element we have

$$\begin{aligned} I_{xy} &= -\sum_i m_i x_i y_i = -\sum_i m_i (R_x + x_i^*)(R_y + y_i^*) \\ &= -\sum_i m_i [R_x R_y + R_x y_i^* + R_y x_i^* + x_i^* y_i^*] \\ &= -M R_x R_y - \sum_i m_i x_i^* y_i^* = -M R_x R_y + I_{xy}^* \end{aligned} \quad ,$$

where the cross terms again vanish. All other components of $\boldsymbol{I}$ can be treated similarly to give the result

$$\boldsymbol{I} = \boldsymbol{I}_{Translation} + \boldsymbol{I}^* \quad ,$$

where $\boldsymbol{I}_{Translation}$ is given more explicitly as

$$\boldsymbol{I}_{Translation} = \begin{pmatrix} M(R_y^2 + R_z^2) & -M R_x R_y & -M R_x R_z \\ -M R_y R_x & M(R_x^2 + R_z^2) & -M R_y R_z \\ -M R_z R_x & -M R_z R_y & M(R_x^2 + R_y^2) \end{pmatrix} \quad .$$

EULER EQUATIONS OF MOTION

If a rigid body moves with respect to the coordinate system, it is clear that in general the elements of $\boldsymbol{I}$ will change with time. Therefore, once we have found a principal axis system the only way to guarantee that we remain in it is to fasten the coordinate system to the rigid body so that it moves with the body. Such a frame of reference is called a *body fixed frame of reference.* We denote the three coordinate directions of a body fixed frame by unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ in contrast to the three coordinate directions $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ of a space fixed inertial frame. In a body fixed principal axis system we write the relation between $\boldsymbol{L}$ and $\boldsymbol{\omega}$ in the simpler form

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \quad .$$

Here the subscripts $1, 2, 3$ refer to components along the three axes $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, and the diagonal elements I_1, I_2, I_3 are called the *principal moments of inertia.*

Up to this point we have considered rotation about a fixed axis. However, there is a general theorem of rigid body mechanics which says that the most general motion of a rigid body is a translation of its centre of mass plus a rotation with angular velocity $\boldsymbol{\omega}$ about an instantaneous axis of rotation. As time passes, this axis of rotation may move with respect to the body unlike our earlier assumption. An important kinematical result is that for any vector quantity $\boldsymbol{B}$, we have the result

$$\left.\frac{d\boldsymbol{B}}{dt}\right|_{inertial} = \left.\frac{d\boldsymbol{B}}{dt}\right|_{body\ fixed} + \boldsymbol{\omega} \times \boldsymbol{B} \quad .$$

Here the rates of change are calculated respectively in the space fixed inertial frame $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ and in the body fixed frame $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$. By using this result we can analyze the Newtonian angular momentum equation of motion in the body fixed frame with considerable simplification. For example, it is natural to analyze the rotation of the Earth from within the frame which rotates with the Earth. To see how this works recall that for a body that is torque-free we have

$$\left.\frac{d\boldsymbol{L}}{dt}\right|_{inertial} = 0 \quad .$$

By using the kinematical result above we can re-write this in the body fixed frame as

$$\left.\frac{d\boldsymbol{L}}{dt}\right|_{body\ fixed} + \boldsymbol{\omega} \times \boldsymbol{L} = 0 \quad .$$

However, in the body fixed principal axis system we write $\boldsymbol{L}$ as

$$\boldsymbol{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3) \quad ,$$

where we use the principal moments of inertia. Correspondingly we have

$$\boldsymbol{\omega} \times \boldsymbol{L} = (\omega_2\omega_3(I_3 - I_2), \omega_3\omega_1(I_1 - I_3), \omega_1\omega_2(I_2 - I_1)) \quad ,$$

giving

$$\begin{aligned} \left.\frac{d\boldsymbol{L}}{dt}\right|_{body\ fixed} &= (I_1\dot{\omega}_1, I_2\dot{\omega}_2, I_3\dot{\omega}_3) \\ &= -\boldsymbol{\omega} \times \boldsymbol{L} = (\omega_2\omega_3(I_2 - I_3), \omega_3\omega_1(I_3 - I_1), \omega_1\omega_2(I_1 - I_2)) \end{aligned} \quad .$$

These are called the Euler equations for the angular velocity components and they hold in the rotating body fixed frame. We may write them as

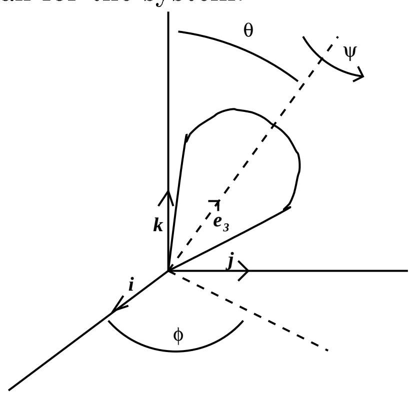
$$\begin{aligned} I_1\dot{\omega}_1 &= \omega_2\omega_3(I_2 - I_3) \quad , \\ I_1\dot{\omega}_2 &= \omega_3\omega_1(I_3 - I_1) \quad , \\ I_1\dot{\omega}_3 &= \omega_1\omega_2(I_1 - I_2) \quad . \end{aligned}$$

More generally, if there were a torque $\boldsymbol{N} = (N_1, N_2, N_3)$ acting on the rigid body the Euler equations simply have an additional term on the right hand side,

$$\begin{aligned} I_1\dot{\omega}_1 &= \omega_2\omega_3(I_2 - I_3) + N_1 \quad , \\ I_1\dot{\omega}_2 &= \omega_3\omega_1(I_3 - I_1) + N_2 \quad , \\ I_1\dot{\omega}_3 &= \omega_1\omega_2(I_1 - I_2) + N_3 \quad . \end{aligned}$$

LAGRANGIAN EQUATIONS FOR A ROTATING TOP

The Euler equations above are a re-expression of the Newtonian angular momentum equation. To get a true Lagrangian description of rotation we need appropriate generalised coordinates to describe the orientational degrees of freedom of the rigid body. These coordinates are the *Euler angles.* To make things simple, let us consider a body with an axis of rotational symmetry which has one point on this axis fixed, for example a pivot point. We can use the Euler angles to express the kinetic energy and thereby get a Lagrangian for the system.



THE LAGRANGIAN

We take the $\boldsymbol{e}_3$ axis to be the axis of rotational symmetry: it is sometimes called the *figure axis.* With an axis of rotational symmetry two of the principal moments of inertia are equal, $I_1 = I_2$ while the third, I_3, is unequal to these two. In the body fixed frame we can write the angular momentum and angular velocity as

$$\begin{aligned} \boldsymbol{L} &= I_1(\omega_1 \boldsymbol{e}_1 + \omega_2 \boldsymbol{e}_2) + I_3 \omega_3 \boldsymbol{e}_3 \\ \boldsymbol{\omega} &= \omega_1 \boldsymbol{e}_1 + \omega_2 \boldsymbol{e}_2 + \omega_3 \boldsymbol{e}_3 \end{aligned} \quad ,$$

so the the kinetic energy can be written as

$$T = \frac{1}{2} I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2 = \frac{1}{2} I_1(\boldsymbol{\omega}^2 - \omega_3^2) + \frac{1}{2} I_3 \omega_3^2 \quad .$$

We can easily express $\boldsymbol{\omega}$ in terms of the rates of change of the Euler angles θ, ϕ, ψ. Define the unit vector

$$\boldsymbol{n} = \frac{\boldsymbol{k} \times \boldsymbol{e}_3}{|\boldsymbol{k} \times \boldsymbol{e}_3|} = \frac{\boldsymbol{k} \times \boldsymbol{e}_3}{\sin\theta} \quad ,$$

which defines the axis of rotation if the angle θ increases with ϕ and ψ held fixed. The total angular velocity is then just a sum of three independent contributions, one for each of the Euler angles,

$$\boldsymbol{\omega} = \dot{\theta} \boldsymbol{n} + \dot{\phi} \boldsymbol{k} + \dot{\psi} \boldsymbol{e}_3 \quad .$$

We then find the kinetic energy by calculating

$$\begin{aligned} \omega_3 &= \boldsymbol{\omega} \cdot \boldsymbol{e}_3 = \dot{\phi} \boldsymbol{k} \cdot \boldsymbol{e}_3 + \dot{\psi} = \dot{\phi} \cos\theta + \dot{\psi} \quad , \\ \omega_3^2 &= \dot{\phi}^2 \cos^2\theta + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi} \cos\theta \quad , \\ \boldsymbol{\omega}^2 &= \boldsymbol{\omega} \cdot \boldsymbol{\omega} = \dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi} \boldsymbol{k} \cdot \boldsymbol{e}_3 = \dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi} \cos\theta \quad . \end{aligned}$$

Finally,

$$\boldsymbol{\omega}^2 - \omega_3^2 = \dot\theta^2 + \dot\phi^2(1-\cos^2\theta) = \dot\theta^2 + \dot\phi^2\sin^2\theta \quad .$$

Putting this together gives the kinetic energy as

$$T = \frac{1}{2}I_1(\dot\theta^2 + \dot\phi^2\sin^2\theta) + \frac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2 \quad .$$

If we have a uniform gravitational field acting in the negative $\boldsymbol{k}$ direction there will also be a potential energy

$$V = Mg\ell\cos\theta \quad ,$$

where ℓ is the distance from the piviot to the centre of mass of the top. The full Lagrangian is then

$$L = \frac{1}{2}I_1(\dot\theta^2 + \dot\phi^2\sin^2\theta) + \frac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2 - Mg\ell\cos\theta \quad .$$

THE LAGRANGE EQUATIONS

We see by inspection of L that ϕ and ψ are cyclic variables so that there must be two corresponding conserved generalised momenta p_ϕ and p_ψ. The angle θ is not cyclic so we have for the first lagrange equation

$$I_1\ddot\theta - I_1\dot\phi^2\sin\theta\cos\theta + I_3(\dot\phi\cos\theta + \dot\psi)\dot\phi\sin\theta - Mg\ell\sin\theta = 0 \quad .$$

The next two equations are simply

$$\frac{dp_\phi}{dt} = 0 \quad , \quad \frac{dp_\psi}{dt} = 0 \quad ,$$

where the generalised momenta are

$$p_\phi = I_1\dot\phi^2\sin^2\theta + I_3(\dot\phi\cos\theta + \dot\psi)\cos\theta = constant$$
$$p_\psi = I_3(\dot\phi\cos\theta + \dot\psi) = constant \quad .$$

These two momenta have a simple physical interpretation. We can show that

$$p_\phi = \boldsymbol{L}\cdot\boldsymbol{k} = L_z \quad , \quad p_\psi = \boldsymbol{L}\cdot\boldsymbol{e}_3 = L_3 \quad .$$

Thus two components of $\boldsymbol{L}$ are conserved, the component along the space fixed $\boldsymbol{k}$ axis, L_z, and the component along the body fixed $\boldsymbol{e}_3$ axis, L_3.