SUMMARY OF NEWTONIAN MECHANICS

BASIC NOTATION

m_i is the mass of particle i, $i = 1, \ldots, N$

$\boldsymbol{r}_i(t)$ is the position of particle i at time t

$\boldsymbol{F}_i$ is the net force on particle i

$$\boldsymbol{F}_i = \boldsymbol{F}_i^e + \sum_{j \neq i} \boldsymbol{F}_{ij}$$

where $\boldsymbol{F}_i^e$ is the *external* force on particle i, that is, the force from outside the system of particles, while $\boldsymbol{F}_{ij}$ is the internal force of particle j ON particle i.

TRANSLATIONAL MOMENTUM

M is the total mass of the system

$$M = \sum_{i=1}^{N} m_i \quad .$$

The centre of mass position $\boldsymbol{R}$ is defined by

$$\boldsymbol{R} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{1}{M} \sum_{i=1}^{N} m_i \boldsymbol{r}_i \quad .$$

The momentum of particle i is defined by

$$\boldsymbol{p}_i = m_i \boldsymbol{v}_i = m_i \frac{d\boldsymbol{r}(t)}{dt} \quad ,$$

and the total momentum of the system is defined as

$$\boldsymbol{P}_{tot} = \sum_{i=1}^{N} \boldsymbol{p}_i = \sum_{i=1}^{N} m_i \boldsymbol{v}_i = M\dot{\boldsymbol{R}} = M \frac{d\boldsymbol{R}}{dt} \quad .$$

EQUATION OF MOTION FOR TRANSLATIONAL MOMENTUM

If the internal forces obey the weak form of the Third Law then we have for the total momentum

$$\frac{d\boldsymbol{P}_{tot}}{dt} = \dot{\boldsymbol{P}}_{tot} = \boldsymbol{F}_{tot}^e \quad ,$$

where $\boldsymbol{F}_{tot}^e$ is the net external force on the system

$$\boldsymbol{F}_{tot}^e = \sum_{i=1}^{N} \boldsymbol{F}_i^e \quad .$$

ANGULAR MOMENTUM

the angular momentum of particle i, $\boldsymbol{L}_i$, is defined by

$$\boldsymbol{L}_i = \boldsymbol{r}_i \times \boldsymbol{p}_i$$

and depends on the choice of origin of coordinates. The total angular momentum of the system is given by

$$\boldsymbol{L}_{tot} = \sum_{i=1}^{N} \boldsymbol{L}_i = \sum_{i=1}^{N} \boldsymbol{r}_i \times \boldsymbol{p}_i \quad .$$

TORQUE

The net torque on particle i, $\boldsymbol{N}_i$, is defined to be

$$\boldsymbol{N}_i = \boldsymbol{r}_i \times \boldsymbol{F}_i \quad ,$$

and also depends on the choice of origin.

EQUATION OF MOTION FOR ANGULAR MOMENTUM

If the internal forces satisfy the strong form of the Third Law (Central two-body forces) then we have

$$\frac{d\boldsymbol{L}_{tot}}{dt} = \dot{\boldsymbol{L}}_{tot} = \boldsymbol{N}^e_{tot} \quad ,$$

where $\boldsymbol{N}^e_{tot}$ is the net external torque on the system

$$\boldsymbol{N}^e_{tot} = \sum_{i=1}^{N} \boldsymbol{N}^e_i \quad .$$

CENTRE OF MASS FRAME

The centre of mass reference frame is one whose origin of coordinates is chosen to lie at the centre of mass position $\boldsymbol{R}$ defined above. Quantities defined in this frame of reference are denoted with an asterisk. Thus $\boldsymbol{r}^*_i$ is the position of particle i with respect to the centre of mass. The transformation equation between a general inertial frame and the centre of mass frame (not necessarily inertial) is given by

$$\boldsymbol{r}_i = \boldsymbol{R} + \boldsymbol{r}^*_i \quad .$$

In the centre of mass frame we can prove that, by its definition, we always have

$$\boldsymbol{P}^*_{tot} = 0 \quad .$$

The angular momentum can be expressed as

$$\boldsymbol{L}_{tot} = \boldsymbol{R} \times M\dot{\boldsymbol{R}} + \boldsymbol{L}^*_{tot} = \boldsymbol{R} \times \boldsymbol{P}_{tot} + \boldsymbol{L}^*_{tot} \quad .$$

We can show that in the centre of mass frame we always have the equation of motion

$$\frac{d\boldsymbol{L}^*_{tot}}{dt} = \dot{\boldsymbol{L}}^*_{tot} = \boldsymbol{N}^{e\,*}_{tot} \quad ,$$

where $\boldsymbol{L}^*_{tot}$ and $\boldsymbol{N}^{e\,*}_{tot}$ are respectively the total angular momentum calculated in the centre of mass frame and the net external torque, also calculated in the centre of mass frame. This equation holds even if the centre of mass frame itself is not an inertial frame.

ENERGY CONSERVATION

For energy conservation to hold in Newtonian mechanics we need to specify further properties of the forces that act. Specifically we need to require that the forces are *conservative fields.* A conservative force field is a position dependent field $\boldsymbol{F}(\boldsymbol{r})$ such that

a) the curl of $\boldsymbol{F}$ vanishes, $\boldsymbol{\nabla} \times \boldsymbol{F}(\boldsymbol{r}) = 0$,

b) or equivalently the line integral of $\boldsymbol{F}$ between any two points is path independent and depends only on the end points of integration.

Such a vector field can be replaced by a scalar field $V(\boldsymbol{r})$ called the potential of the field of force which is defined up to an additive constant so that

$$V(\boldsymbol{r}_b) - V(\boldsymbol{r}_a) = -\int_{\boldsymbol{r}_a}^{\boldsymbol{r}_b} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} \quad .$$

Equivalently, we can find the force field from the gradient of the potential function

$$\boldsymbol{F}(\boldsymbol{r}) = -\nabla V(\boldsymbol{r}) \quad .$$

We can regard the potential energy as stored work in the sense that $V(\boldsymbol{r}_b) - V(\boldsymbol{r}_a)$ is the work done against the field $\boldsymbol{F}(\boldsymbol{r})$ in moving the particle from $\boldsymbol{r}_a$ to $\boldsymbol{r}_b$.

TOTAL POTENTIAL ENERGY

For a system of particles whose internal and external forces are conservative fields, we can define the total potential energy of the system as

$$V(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = \sum_{i=1}^{N} V^e(\boldsymbol{r}_i) + \sum_{i<j} V^e_{ij}(\boldsymbol{r}_{ij}) \quad ,$$

where $V^e(\boldsymbol{r})$ is the potential energy associated with the external force field $\boldsymbol{F}^e(\boldsymbol{r})$ and $V^e_{ij}(\boldsymbol{r}_{ij})$ is the potential energy associated with the internal two particle force $\boldsymbol{F}_{ij}(\boldsymbol{r}_{ij})$ where $\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$. The potential energy of each pair must be counted once and once only in the total potential energy. That is the reason for the restriction $i < j$ in the sum above. Since $V_{ij} = V_{ji}$ and $V_{ii} = 0$ (no particle can exert a force on itself), we alternatively can write an unrestricted sum which counts all pairs twice and then divide by two to avoid double counting

$$V(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = \sum_{i=1}^{N} V^e(\boldsymbol{r}_i) + \frac{1}{2} \sum_{i,j=1}^{N} V^e_{ij}(\boldsymbol{r}_{ij}) \quad .$$

TOTAL KINETIC ENERGY

The total kinetic energy is defined as

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \boldsymbol{v}_i^2 \quad .$$

THE ENERGY EQUATION

We obtain the energy equation by integrating Newton's equations of motion for all N particles between an initial time and initial set of positions and a final time and final set of positions. We deduce from this that

$$E_{initial} = T_{initial} + V_{initial} = T_{final} + V_{final} = E_{final} \quad .$$

THE VIRIAL THEOREM

A result which is closely related to the concept of energy is known as the virial theorem and is due to Clausius, a German physicist who also contributed much to thermodynamics. This theorem is derived by considering the quantity

$$G = \sum_{i=1}^{N} \boldsymbol{p}_i \cdot \boldsymbol{r}_i = \sum_{i=1}^{N} m_i \boldsymbol{v}_i \cdot \boldsymbol{r}_i \quad .$$

Using Newton's second law for each particle we can show that

$$\frac{dG}{dt} = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \boldsymbol{r}_i + 2T \quad ,$$

where $\boldsymbol{F}_i$ is the net force on particle i and T is the total kinetic energy of the system. We define a time average over time interval τ of any mechanical quantity that depends on position and velocities $f(\boldsymbol{r}_i, \boldsymbol{v}_i)$ by

$$\overline{f(\boldsymbol{r}_i, \boldsymbol{v}_i)} = \frac{1}{\tau} \int_0^{\tau} f(\boldsymbol{r}_i, \boldsymbol{v}_i) dt.$$

Time averaging the previous equation gives

$$\frac{G(\tau) - G(0)}{\tau} = \overline{\sum_{i=1}^{N} \boldsymbol{F}_i \cdot \boldsymbol{r}_i} + 2\overline{T} \quad .$$

If the motion is such that it is periodic or else all positions and velocities are bounded, then as $\tau \to \infty$ the left hand side vanishes to give the virial theorem

$$\overline{T} = -\frac{1}{2} \overline{\sum_{i=1}^{N} \boldsymbol{F}_i \cdot \boldsymbol{r}_i} \quad .$$

This theorem is of great use in investigating the gravitating mass within a galaxy or in computing pressures in computer simulations of gases and liquids.