

## Simultaneity and Time Dilation

We derive the equations of Special Relativity from two Postulates:

1. **The Principle of Relativity.** All the laws of physics have the same form in all inertial frames of reference. This is an extension of Newton's principle, since he knew only that all *mechanical* laws are the same in all inertial frames. In particular, we are extending it to the laws of electromagnetism.
2. **The Principle of the Constancy of the Speed of Light.** The speed of light in vacuum has the same value,  $\sim 3 \times 10^8 \text{ m s}^{-1}$ , in all inertial frames, irrespective of the velocity of the observer or the velocity of the source emitting the light. This does in fact follow from 1. Maxwell's equations of electromagnetism yield  $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$  for the speed of light in a vacuum.

Hence if 1 is true it follows that 2 must be.

**SIMULTANEITY:** How do we measure time? In order to set up a universal time scale we must give meaning to

**“EVENTS A AND B OCCUR AT THE SAME TIME.”**

Firstly, take one inertial frame and set up a time scale at one point. That is, put a clock there. Look at the clock when an event occurs. The time of an event at that point is the clock reading  $t$  when the event occurs. Two events A and B at that point are simultaneous if  $t_A = t_B$ .

If the two events occur at different places, we need a clock in each place. The time of each event is the reading of the clock at the same place as the event. So we have to read two clocks, and so we must previously *synchronise* the clocks.

Some obvious ways to do this are obviously wrong:

- a) Set the clocks so that they read the same as viewed from A. This neglects the fact that it takes time for light to get from B to A. **Wrong!**
- b) Set both clocks at A and then move one clock to B. This neglects the fact that clocks are always perturbed by movement – accelerations and decelerations – e.g. a pendulum clock runs faster while it is accelerated. **Wrong!**

*Answer:* We use our knowledge of the speed of light and set the clock at B so that viewed from A it is a bit slow, reading  $t_A - L/c$ , if  $L$  is the distance to B. In this way we can install *synchronised* clocks throughout space.

*Then events occurring at different places are simultaneous when the synchronised clocks in the different places read (or record) the same time for the events.*

**BUT** – *We do not KNOW that events which are simultaneous in this inertial frame would be regarded by an observer in another inertial frame moving at relative speed  $v$  as simultaneous according to his own synchronised clocks. That would be an assumption about the way the world is, rather than the outcome of doing physics. And the assumption would be **wrong**.*

**Example:** Lightning strikes both ends of a moving train (inertial frame  $S'$ ). There is an observer  $O'$  in the observation carriage at the centre of the train, and an observer  $O$  standing by the tracks (inertial frame  $S$ ).

Let  $O$  observe that his clocks read the same time for the two lightening strikes. He concludes that the strikes are simultaneous.

Observer  $O'$  is moving with the train, towards the light coming from the front strike A and away from the light coming from the rear strike B. He meets the light from the front sooner, and the light from the rear reaches him later. He concludes that the two strikes are not simultaneous, that  $t_A < t_B$ .

Which observer is right? The answer is that BOTH ARE RIGHT. BOTH HAVE MEASURED THE TIMES USING THE SAME PHYSICS. The concept of simultaneity is *not absolute* and therefore *Time is not absolute*.

**TIME DILATION:** How do we measure time? We construct clocks consisting of instruments which – according to the laws of physics – carry out some cyclical operation which – according to the laws of physics – takes the same time each time.

Let us construct a clock by bouncing a pulse of light between two plane parallel mirrors at separation  $d$ . (We might have laser amplification, but we can ignore that refinement). The round-trip time is

$$T_0 = \frac{2d}{c}$$

and that is the period of this clock. (E.g. for  $d = 1\text{m}$ , the period is  $6\frac{2}{3}\text{ ns}$ , and the frequency of this clock is  $150\text{ MHz}$ .)

Put this clock vertically in the train, and for  $O'$  it works perfectly, It obeys the laws of physics and keeps good time. Viewed by  $O$ , however, the clock period is  $T$ , and the train moves a distance  $vT$  in a round-trip, and so the distance the light goes is not  $2d$  but (by Pythagoras)

$$L = 2\sqrt{d^2 + (\frac{1}{2}vT)^2}$$

He will observe that the period  $T$  is not  $T_0$  but

$$T = \frac{L}{c} = \frac{2}{c}\sqrt{d^2 + (\frac{1}{2}vT)^2}$$

But  $d = \frac{1}{2}cT_0$ , so

$$\begin{aligned} T^2 &= \frac{4}{c^2} \left( \frac{1}{4}c^2T_0^2 + \frac{1}{4}v^2T^2 \right) = T_0^2 + \frac{v^2}{c^2}T^2 = T_0^2 + \beta^2T^2 \\ T^2(1 - \beta^2) &= T_0^2 \\ T &= T_0 \frac{1}{\sqrt{1 - \beta^2}} \end{aligned}$$

This is **Time Dilation**. The observer in  $S$  sees the clock in  $S'$  as running slow. Note that the analysis is symmetrical. The observer in  $S'$  sees the clock in  $S$  as running slow.

Notation:  $\beta = \frac{v}{c}$  and  $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \beta^2}}$  occur so often in

Relativity that we use the symbols  $\beta$  and  $\gamma$  in this way a lot.

Thus, e.g.,  $T = \gamma T_0$

**PROPER TIME:** How do we measure the time interval between two events? *Proper Time* is defined as the time interval between two events measured by an observer for whom the two events occur at the same place.

In the example of the optical clock in the train, the proper time is  $T_0$ , measured in  $S'$ .

The proper time is always the *shortest* time interval between two events. Any observer for whom the events occur in different places measures a longer interval ( $\gamma > 1$ ). *Time dilates. Time does not contract.*

**Experimental Example: Muon Lifetime.** Muons are subatomic particles with a half-life (measured at rest in the laboratory frame) of  $2.2\mu\text{s}$ . They are produced in the upper atmosphere (several kilometres high) in collisions between cosmic rays and air atoms, with very high energy, and velocities close to  $0.99c$  with respect to the Earth. Without time dilation, they would travel a distance  $s = 0.99c \times 2.2 \times 10^{-6} = 650\text{ m}$  in a half-life. We would detect very few at the surface of the Earth.

For  $\beta = 0.99$ ,  $\gamma \sim 7$ . So we expect them to travel  $7 \times 650\text{m} = 4.5\text{ km}$  in a half-life, and to reach the Earth's surface in large quantities (as observed).

**Theoretical Example: A Spaceship** travels at  $0.6c$  relative to the Earth ( $\beta = 0.6$  and  $\gamma = 1\frac{1}{4}$ ). As it passes Earth, timers are started both on Earth and on the spaceship. We ask (a) when Earth says the spaceship is  $0.9$  million km away, what does the spaceship timer say? (b) when the timer on the spaceship reads  $4\text{s}$ , what does the timer on Earth read?

*Identify Problem:* These questions are badly posed and ambiguous. Let  $S$  be the Earth's frame of reference and  $S'$  be that of the spaceship. To clarify the questions, we identify the events and target variables that are being asked about.

a) The two events are the spaceship passing the Earth, and the spaceship passing the milestone at  $9 \times 10^8\text{ m}$  away. The target variable is the time interval between these events in  $S'$

b) The three events are (A) the spaceship passing the Earth, (B) the spaceship's timer reaching  $4\text{s}$ , and (C) the Earth's timer read. The target variable is the time interval between A and C in  $S$ . B and C are simultaneous in  $S'$ .

*Calculation:* For (a), the two events occur at different places in  $S$  and the same place in  $S'$ . The interval  $T = L/v = 9 \times 10^8 / 1.8 \times 10^8 = 5\text{ s}$  is therefore *not* the proper time, it is  $T = \gamma T_0$ . Therefore, the proper time  $T_0$  measured in  $S'$  is  $T/\gamma = 4\text{ s}$ .

For (b), it is tempting – but **wrong** – to use the results from (a) and answer  $5\text{ s}$ . The events A and C do not happen in the same place in  $S'$  so the interval of  $4\text{s}$  is  $\gamma$  times the proper time  $T_0$  – which is therefore  $3.2\text{ s}$ .

**THE TWIN PARADOX:** This has caused more confusion and doubt among philosophers – and even some scientists – than any other aspect of Relativity. *The confusion is due to failure to clarify the events.*

Consider a pair of twins A and B. A stays on Earth, while B takes a spaceship travelling at  $0.5c$  to a planet 10 light-years away ( $\beta = 0.5$  and  $\gamma = 2/\sqrt{3} = 1/0.866$ ). Immediately on arrival he turns round and comes back. For A, B's journey lasts 40 years and A will have aged 40 years when his twin B returns. A knows that the proper time on the spaceship for the outward journey is  $20/\gamma = 17.3$  years. And the proper time for the return journey is also 17.3 years. So B has aged only 34.6 years. B is now younger than his twin A.

The Paradox, the confusion, arises from the argument: "Motion is relative, so we may consider what happens from B's point of view. He sees the Earth rush away from him while he is at rest. Then the Earth comes rushing back. So he may argue that his twin, A, should be younger than he is.

This appears to be a contradiction at the heart of Special Relativity. Just Google e.g. Einstein and Dingle (a physicist and philosopher), or go directly to <http://www.mathpages.com/home/kmath024/kmath024.htm> to see some of the trouble it has caused.

The *general* answer is that Special Relativity deals only with *inertial* frames moving relative to each other with *uniform velocity*. Twin B has undergone acceleration while Twin A has not, and that makes the situation not symmetrical. Twin B has been in *two* inertial frames. So the paradox says nothing about Special Relativity.

That answer is not sufficient to clear up all the confusion. Analyses of particular situations can help. Let a clock on Earth emit a powerful flash of light once a year ( $T_0$ ), and let Twin B observe these flashes from his spaceship. B knows that in his frame of reference, these flashes are emitted once every  $T = \gamma T_0 = 1.15$  years. In addition, the Earth is receding at  $0.5c$ , so each flash travels  $0.5 \times 1.15c$  light-years further than the one before, which takes 0.575 years. So he sees them at intervals 1.73 years. Since his outward journey takes 17.3 years, he sees **ten flashes**. On the return journey, he sees the flashes every  $1.15 - 0.575$  years = 0.58 years. During the return, 17.3 years, he thus sees **thirty flashes**. Total 40, so he knows that 40 years have passed on Earth. *Twin B has correctly observed time dilation on Earth relative to his frames of reference. He correctly observes 40 years passing on Earth. He knows that A will be older than him. There is no contradiction.*