Section 1.

Dynamics (Newton's Laws of Motion)

Two approaches:

1) Given all the forces acting on a body, *predict* the subsequent (changes in) motion.
2) Given the (changes in) motion of a body, *infer* what forces act upon it.

Review of Newton's Laws:

First Law: A body at rest remains at rest,
a body in motion continues to move at constant velocity,
unless acted upon by an external force.

Note: This is only true in an inertial frame.
Example 1: You are on a train with a ball on the floor. The train accelerates. What does the ball do?
Example 2: You observe the world from a rotating carousal. All other objects are changing their velocities wrt you as you turn. What forces must be applied to them to achieve this?

Second Law: A force acting on a body causes an acceleration of the body,
in the direction of the force, proportional to the force,
and inversely proportional to the mass.

Note: This is only true in an inertial frame.

Express as $\vec{a} = \frac{\vec{F}}{m}$, or $\vec{F} = m\vec{a}$, or $\vec{F} = m\frac{d\vec{v}}{dt}$

Example 1: Force parallel to velocity. What does the body do?
Example 2: Force always at right angles to velocity. What does the body do?

We define the linear momentum $\vec{P} = m\vec{v}$, so that

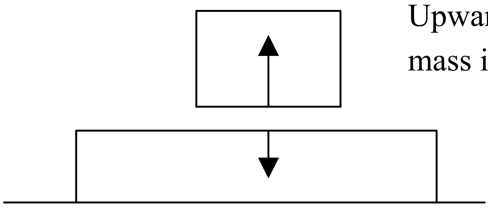
$$\boxed{\vec{F} = \frac{d\vec{P}}{dt}}$$

We have two ways of *measuring* mass:
Inertial method – apply a force and measure acceleration $\Rightarrow m_{inertial}$
Gravitational method – weight the object (no motion) $\Rightarrow m_{grav}$

Later we will see that they are the same.

Third Law: To every action there is an equal and opposite reaction.

Example 1:



Upward force on mass is $\vec{N}$

Downwards force on plinth is $\vec{W} = m\vec{g}$

And $\vec{N} = -\vec{W}$

Example 2: A rocket engine generates the force $\vec{F}_{thrust}$ *and applies the force* F_{gas} *to the exhaust,*

And $\vec{F}_{thrust} = -\vec{F}_{gas}$

Applications of Newton's Laws of Motion

***Example* 1: Inclined Ramp**

An 8kg cart is pulled up a frictionless slope inclined at 20°. Determine the force if the cart is to move

a) With uniform motion,
b) With an acceleration of 0.2 m s^{-2} up the plane.

a) Resolve forces parallel to surface, then $F = 8g\sin 20^\circ = 26.8\text{N}$

b) Hence $F - 8g\sin 20^\circ = ma = 8\times 0.2$, so $F = 28.4\text{N}$

***Example* 1: The Pulley**

A weightless cord hangs over a frictionless pulley. A mass of 1kg hangs at one end of the cord and a mass of 2kg at the other. Calculate

a) the acceleration of the masses,
b) the tension in the cord,
c) the reaction (the upwards force) exerted by the pulley.

Analysis: String must be at constant tension *T* throughout.

Let upwards acceleration be positive.
Let string accelerate at *a* on 1kg side
Then for the 1kg mass,

$$T - mg = ma \text{, i.e. } T - g = a$$

and for the 2kg mass

$$T - mg = -ma \text{ i.e. } T - 2g = -2a$$

a) Eliminating T, we obtain $a = g/3$
b) From either equation, $T = 4/3\ g$
c) Then the reaction is $R = 2T = 8/3\ g$

***Now solve the same problem for masses** m **and** m'*

Equilibrium of a Solid Body

The static equilibrium of a solid body entails two distinct conditions:

1) The net force tending to accelerate it is zero

$$\sum_i \vec{F}_i = 0 \qquad \textit{Condition of Translational Equilibrium}$$

Note: All the forces **do not** have to go through the same point. A ladder leaning against a wall has reaction and friction forces at each end, which do not go through the centre or any other single point.

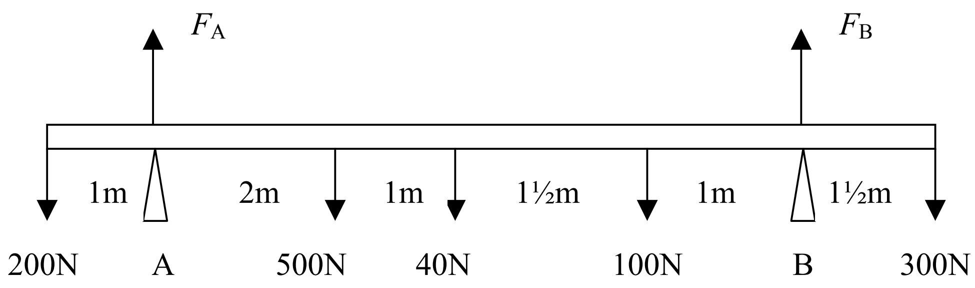
2) The net torque tending to rotate it is zero

$$\sum_i \vec{T}_i = 0 \qquad \textit{Condition of Rotational Equilibrium}$$

Note: A body may be in translational equilibrium while out of rotational equilibrium. It may also be in rotational equilibrium while out of translational equilibrium.

***Example*: A Loaded Bar**

A weightless bar rests on two supports. Several loads are hung from it



Calculate the forces F_A and F_B

Analysis:

1) Translational equilibrium, and note all forces are in *y*-direction. So forces simply sum to zero:

$$\sum_i F_i = 0 = 200 + 500 + 40 + 100 + 300 - F_A - F_B$$

$$F_A + F_B = 1140\text{N}$$

2) Rotational equilibrium, so total moment about any point is zero. Taking moments about A:

$$\sum_i T_i = \sum_i F_i x_i = 0$$

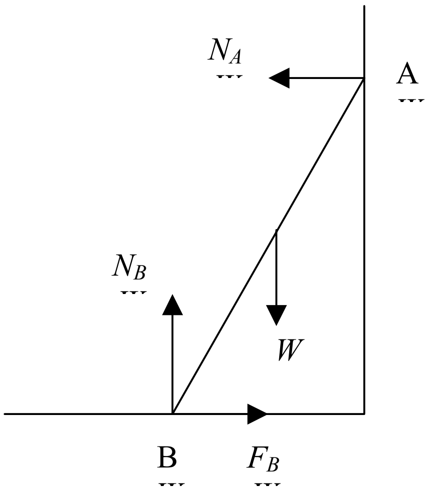
$$= 200 \times -1 - F_A \times 0 + 500 \times 2 + 40 \times 3 + 100 \times 4.5 - F_B \times 5.5 + 300 \times 7$$

$$= 3470 - 5.5 F_B$$

So $F_B = \dfrac{3470}{5.5} = 630.9N$

Therefore $F_A = 1140 - F_B = 509.1N$

Example: **Ladder against Wall**



Assume wall is frictionless, i.e. $F_A = 0$

Let the weight of the ladder be 16 kg, so that $W = 160$ N, and let it be at 60° (π/3 radians) to the ground.

Find the forces acting at each end of the ladder
Find the minimum coefficient of friction against the ground.

Analysis

Translation equilibrium: *Equate forces resolved in two axes*

$F_B = N_A$

$W = N_B = 160$ N

Rotational equilibrium: *Take moments about B*

$W \times \frac{1}{2}L \cos 60^\circ = N_A \times L \sin 60^\circ$

$\Rightarrow \quad N_A = 80 \cot 60^\circ = 46.2$ N

Friction:

$F_B = N_A \leq \mu_S N_B$

$\Rightarrow \quad \mu_S \geq 46.2 / 160 \approx 0.29$

Frictional Forces

Friction is due to interactions between the atoms of an object and those of a surface that it touches. *Microscopic* roughness plays a role too. *Macroscopic* roughness is treated separately.

There are two kinds of frictional forces:

- *Static* Friction, when the surfaces are at rest
- *Kinetic* or Sliding Friction, when the surfaces are in relative motion.

In both cases, the frictional force *opposes* motion between the surfaces, and its magnitude is

$$F = \mu_S N$$

$$F = \mu_K N$$

$$\mu_S \geq \mu_K$$

where μ_S and μ_K are the *coefficients of static / kinetic friction* respectively.

Definitions:

- $F = \mu_S N$ is the *minimum* force required to set in motion the surfaces in contact and initially at rest, when the normal force (*contact* force) is N.
- $F = \mu_K N$ is the *minimum* force required to maintain the relative motion of the surfaces in contact, when the normal force (*contact* force) is N.

***Example*: Inclined Plane with Friction**

Sliding Uphill: Resolving forces parallel to inclined plane,

and using $F = ma$,

$F_{pull} - mg \sin \theta - \mu N = ma$

a is uphill acceleration.

$F_{pull} = mg \sin \theta + \mu mg \cos \theta + ma$

Sliding Uphill: Resolving forces parallel to inclined plane,

and using $F = ma$,

$F_{pull} + mg \sin \theta - \mu N = ma$

a is downhill acceleration

$F_{pull} = - mg \sin \theta + \mu mg \cos \theta +$

$a = 0$ for uniform motion.

Just balancing, $F_{pull} = 0 = a$, then $\mu = \tan \theta$

Acceleration with Varying Mass: The Rocket

A rocket at take-off has maximum mass (payload, structure and fuel) which decreases during flight as fuel is used. This is a characteristic example of a varying mass problem.

Let the rocket operate by ejecting exhaust at nozzle velocity v_e

at the rate $\dfrac{dm}{dt}$ (mass per unit time).

At time t let the rocket have velocity $v(t)$ reltive to an inertial frame and mass $m(t)$. The exhaust mass *dm* that departs in the next time interval *dt* therefore has velocity $v'(t) = v(t) - v_e$ in the inertial frame.

Conserving momentum $p = mv$:

At time *t*, $p(t) = m(t)v(t)$

At time $t + dt$ $p(t+dt) = (m - dm)(v + dv) + v'dm$

which expands to $p(t+dt) = mv + mdv - vdm + vdm - v_e dm$

The momentum is unchanged, so $mdv = v_e dm$

So acceleration is $a(t) = \dfrac{dv(t)}{dt} = \dfrac{v_e}{m(t)} \dfrac{dm(t)}{dt}$

Integrating this from initial conditions, at $t = 0$, $v = 0$, $m = m_0$,

$$v(t) = v_e \log_e \frac{m_0}{m(t)}$$

This is a very important equation for rocket designers. It says that the nozzle velocity must be made as high as possible, and that the fuel must be as high a proportion of the mass as possible. It implies that rockets should be multi-stage.

Work and Power

Work is done when a force acts along a displacement. The work is the energy required to achieve this.

Example: A body mass m falls through a height h. The work done by gravity is $Fh = mgh$.

Only the component of force parallel to the displacement is relevant. If the force is at the angle α to the displacement, the work is $Fs \cos \alpha$. In vector notation, this is

$$W = \vec{F} \cdot \vec{s}$$

If the path is curved, we may want the differential relationship,

$$dW = F \cos\alpha \, dr = \vec{F} \cdot d\vec{r}$$

Power is the rate of doing work.

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v}$$

Kinetic and Potential Energy

Consider $\vec{F} = m\vec{a}$ - a force acting on a particle to accelerate it. The work done is

$$W = \int_A^B dW = \int_A^B \vec{F}.d\vec{r} = m\int_A^B \vec{a}.d\vec{r} = m\int_A^B \frac{d\vec{v}}{dt}.d\vec{r} = m\int_A^B \frac{d\vec{r}}{dt}.d\vec{v}$$

$$= m\int_A^B \vec{v}.d\vec{v} = \tfrac{1}{2}m\left(v_B^2 - v_A^2\right)$$

We define *kinetic energy* accordingly as $\tfrac{1}{2}m\vec{v}.\vec{v} = \tfrac{1}{2}mv^2$

Work done by a force accelerating a particle changes the kinetic energy of the particle.

Work may be done by or against a force due to a field. For example, a charged particle moving in an electric field under the Coulomb force, or a mass moving in a gravitational field.

In these cases we introduce the concept of *potential energy*. The work done *on* the particle by the force is

$$W_{ON} = \int_A^B \vec{F}.d\vec{r} = \int_A^B dE = E_B - E_A$$

The work done *by* the particle is $W_{BY} = -W_{ON} = E_A - E_B$

In differential form,

$$dW = -\vec{F}.d\vec{r} = -dE$$

Potential Energy Curves

We may write potential energy as a function of position, e.g. in one dimension, $E = E(x)$. Then

$$F(x) = -\frac{dE}{dx}$$

is the force acting on the particle. Consequently, *minima* and *maxima* of the potential energy curve are points where $F = 0$ and a stationary particle will remain at rest. The minima are stable (or metastable) and the maxima are unstable.
