General Relativity

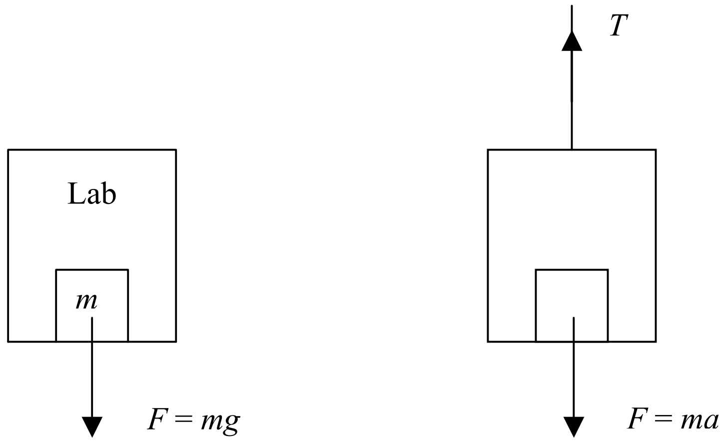
Special Relativity is based on two postulates applicable to inertial frames of reference. We now *generalise* to accelerating frames of reference.

Mass appears in both Newton's gravitational law and his laws of motion.

$$F = \frac{GMm}{r^2}$$

$$F = ma$$

Experimental tests fail to distinguish between the "*m*" in each equation. Einstein proposed that they are in fact the same thing, and therefore that experimental tests within a closed laboratory cannot distinguish between a gravitational field on an inertial laboratory and an accelerating laboratory. They are *equivalent*:



So General Relativity is based on *three* principles:

- **Principle of Relativity**
- **Principle of Constancy of the Speed of Light**
- **Principle of Equivalence**

We shall use these to derive some of the basics of General Relativity, to *curved space* and the *black hole*.

Effect of Gravity on Light: Consider a laser beam shining horizontally across the laboratory which is accelerating vertically with acceleration a. It takes a time L/c to traverse the lab. In that time, the far wall has risen a distance $s = ½\, at^2$

$$s = ½at^2 = ½a\frac{L^2}{c^2}$$

Equivalence says that the light beam will also fall – deflect – in a gravitational field, with g for a. Note that this is the *same* as the prediction for a classical particle with initial horizontal velocity c and KE $½mc^2$ – but the photon has KE of mc^2, for which the classical prediction is half the general relativity prediction. *Furthermore*, the classical photon is the electromagnetic field, which classically doesn't interact with gravity.

Time Dilation in a gravitational field. Consider a laser at the bottom of an accelerating lift emitting light of frequency ν. This light arrives at the ceiling, in time

$$T = \frac{h}{c}$$

and the lift is now travelling upwards at

$$v = a\frac{h}{c}$$

compared with the speed when the light was emitted. So the light is red-shifted (Doppler-shifted) by the factor

$$z \sim \frac{\Delta\lambda}{\lambda} \sim -\frac{\Delta\nu}{\nu} \sim \frac{v}{c} = \frac{ah}{c^2} = \frac{gh}{c^2}$$

where we have used the equivalence principle to say that we get the same red-shift from a gravitational field with $g = a$ as from the acceleration a.

The factor gh is the change in gravitational potential (per unit mass) when we move from the floor to the ceiling of the lift, i.e. $\Delta\Phi = gh$. So we have the red-shift factor z as

$$z = \frac{\Delta\Phi}{c^2}$$

If an observer on the floor of the lift is using the laser as a clock – with a tick frequency of ν - then an observer on the ceiling with an identical laser as her clock sees that the clock on the floor is running slow. This is *gravitational time dilation.*

Curved Space: If gravity causes time dilation, then gravity upsets our Lorentz-invariant interval between two events,

$$s = \sqrt{x^2 + y^2 + z^2 - c^2t^2}$$

Pythagoras is always upset on a curved surface – e.g. on the surface of a sphere, $s < \sqrt{x^2 + y^2}$ where x is the distance East and y is the distance North. This is what we mean by a curved space. General Relativity has a curved four-dimensional *spacetime*.
Mass curves spacetime, and the curvature of spacetime tells masses how to move.

Black Holes: The French mathematician Laplace first speculated about the existence of an object so heavy and compact that the escape speed would be greater than the speed of light.

We had previously, that the escape velocity would be the velocity at which the KE equals the gravitational PE,

$$\tfrac{1}{2}mv^2 = \frac{GMm}{r}$$

Putting $v = c$ gives

$$\frac{GM}{r} = \tfrac{1}{2}c^2$$

The Sun: We may take constants from

http://ssd.jpl.nasa.gov/astro_constants.html

G = 6.67259 (± 0.00030) x 10^{-11} kg^{-1} m^3 s^{-2}

or

Heliocentric gravitational constant	k^2 AU^3 d^{-2} = GM_{sun} =1.32712440018 x 10^{20} (± 8 x 10^9) m^3 s^{-2}

and find that the escape velocity from the surface of the Sun would be the speed of light if

$$r = \frac{2GM}{c^2} = \frac{2 \times 1.33 \times 10^{20}}{\left(3 \times 10^8\right)^2} = 3 \times 10^3 = 3\,\text{km}$$

i.e. if the Sun was compressed to 3 km radius.

The general relativity calculation may be done very approximately by taking $z = 1$, which gives

$$\frac{GM}{r} = c^2$$

which is essentially the same. In fact the exact general relativistic calculation does include the factor of 2.

See websites, e.g.
http://cassfos02.ucsd.edu/public/tutorial/GR.html
for more information, or do the 2nd-year course *Space Time and Gravity.*

END