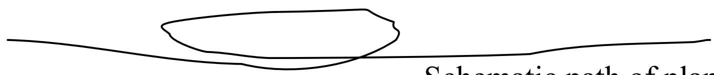
<u>Section 4 Gravity and the Solar System</u>

The oldest "common-sense" view is that the earth is stationary (and flat?) and the stars, sun and planets revolve around it. This GEOCENTRIC MODEL was proposed explicitly by Ptolemy of Alexandria in the 2nd century AD BC.

However, as early as *circa* 270 BC, Aristarchus of Samos had proposed that the Earth revolved round the Sun. This HELIOCENTRIC MODEL did not gain widespread agreement – Archimedes among others did not accept it. So the Geocentric Model prevailed until COPERNICUS (1473 – 1543) revived the Heliocentric Model as a means of resolving the problem of planetary *Retrograde Motion*.



Schematic path of planet in sky

Galilei GALILEO (1564-1642), using telescopes for the first time to view the heavens, observed the phases of Venus and the Moons of Jupiter, which both supported the Heliocentric Model.

Johannes KEPLER (1571-1630) used experimental data on planetary orbits obtained by Tycho BRAHE (1546-1601) to make a theoretical analysis. He deduced three Laws (simply from observation, not from any underlying theory).

Kepler's Laws

1. The planets move on elliptical orbits with the Sun at one focus of the ellipse.
2. The planet sweeps out equal areas in equal times. (While called the Law of Areas, it is about the variations of speed of the planet around its orbit).
3. The square of the period of a planet (the length of its year) is proportional to the cube of the radius of its orbit.

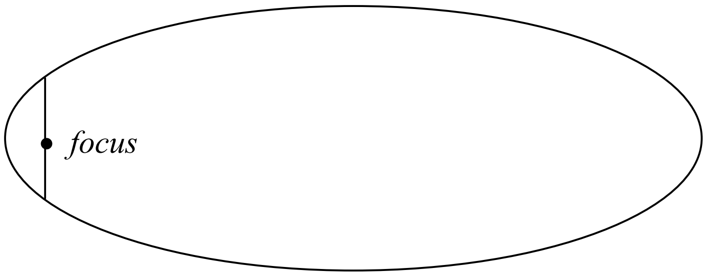
Regarding **Law No.1**, note that an ellipse is characterised by is *semi-major* axis a, by its *semi-minor* axis b, and it eccentricity e. These are related by

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

For a circular orbit, $a = b$ and $e = 0$.

The orbit has a furthest point from the Sun, called the *Aphelion* and a nearest point, the *Perihelion*. For an orbit around the Earth, these are called the *Apogee* and *Perigee*. The major axis joins these points.

For the Earth, $e = 0.017$. For Venus it is still smaller, 0.007. Only for Pluto (perhaps not a real planet?) it is as high as 0.248. For comets, it is close to 1.



Centred at origin: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Polar, origin at focus: $r(1+e\cos\theta)=l$

l is the *semilatus rectum*, $l=\frac{b^2}{a}$

Regarding **Law No.2**, note that the area swept out in a short time *dt* is

$$dA = ½r^2 d\theta$$

So

$$\frac{dA}{dt} = \text{constant} = ½r^2\frac{d\theta}{dt} = ½\omega r^2 = ½vr$$

That is,

$$L = mvr = \text{constant}$$

and the torque,

$$\tau = \frac{dL}{dt} = 0$$

If the torque is zero, any force is a ***central force***.

Regarding **Law No.3**, we shall see later that the derivation of $P^2 \propto r^3$ is very straightforward for a circular orbit with inverse-square gravitation. It is less easy for elliptical orbits, for which the semi-major axis *a* is the key parameter replacing the radius: $P^2 \propto a^3$

Law of Universal Gravity

Many workers tried to find what force-law would produce motion according to Kepler's Laws. Isaac NEWTON (1643 – 1727) established that an INVERSE-SQUARE LAW would do it – that is, he found that the radial acceleration in a Keplerian orbit varies as the inverse square of the distance to the sun. Using his mechanics, $F = ma$, he had at once the inverse-square force-law.

Newton was still a long way from establishing universal gravitation as the cause. For this, he needed a theoretical breakthrough and an experimental verification.

Theoretical Breakthrough: Newton was the first to realise that the Laws of Physics which are established on Earth might also apply in the heavens – in what the Ancients had called the superlunary part of the Universe. Then his inverse-square force might be the force familiar on Earth known as gravity. He knew the acceleration due to gravity at the surface of the Earth – say, 10 m s^{-1}, and the radius of the Earth – say, 6380km. He could calculate the centripetal acceleration of the Moon towards the Earth in its orbit:

$$a_M = \frac{v^2}{r} = r\omega^2 = 384400\ km \times \left(\frac{2\pi}{27.3\ \text{days} \times 24 \times 60 \times 60}\right)^2$$

$$= \frac{3.84 \times 10^8 \times 4\pi^2}{5.56 \times 10^{12}} = 2.73 \times 10^{-3}\ \text{m s}^{-2}$$

So his crucial test was whether the accelerations go as the ratio of the squares of the distances:

$$\frac{r_E^2}{r_M^2} = 2.78 \times 10^{-4}$$

$$\frac{a_M}{a_E} = 2.76 \times 10^{-4}$$

This beautiful confirmation enabled him to propose the Law of Universal Gravitation,

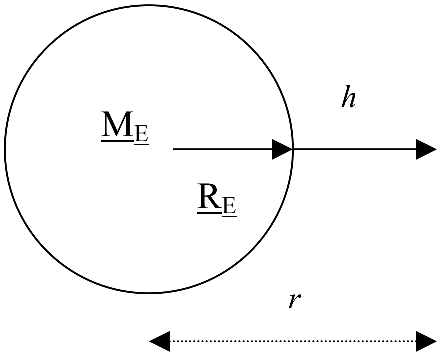
$$F = \frac{GMm}{r^2}$$

Newton could only make rough estimates of the value of G, the Universal Gravitational Constant, as the mass of the Earth could only be estimated. Finding G more accurately is an important experiment that has often been called "Weighing the Earth." See the torsion balance experiment of Henry CAVENDISH (1731 – 1810). It is still the least accurate of all the fundamental constants, being

$$G = (6.6742 \pm 0.0010) \times 10^{-11}\ \text{N m}^2\ \text{kg}^{-2}$$

This makes gravitation ***very weak***. The ratio between the gravitational attraction and elecrostatic repulsion between a pair of protons is 0.81×10^{-36}.

Gravity above the Earth's Surface.



Altitude h
$M_E = 5.97 \times 10^{24}$ kg
$R_E = 6.380 \times 10^{6}$ m
$G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}

$$F = mg = \frac{GMm}{r^2}$$

$$\therefore g = \frac{GM_E}{(R_E + h)^2}$$

$$= 9.81\text{ms}^{-2}\text{at } h = 0$$

$$= 8.18\text{ms}^{-2}\text{at } h = 600\text{km}$$

Derivation of Kepler's 3rd Law:

Consider a satellite in a circular orbit at radius r around a planet of mass M

Centripetal force = gravitational force, i.e. $\dfrac{mv^2}{r} = \dfrac{GMm}{r^2}$

Introduce the *period* $T = \dfrac{2\pi r}{v}$, then $\dfrac{GM}{r^2} = \dfrac{4\pi^2 r}{T^2}$

Rearranging, $T^2 = \dfrac{4\pi^2}{GM} r^3$. *This is Kepler's 3rd Law for the special case of a circular orbit.* The calculation for an elliptical orbit can be done, and yield the same expression with r replaced by the semi-major axis, a.

Gravitation Potential Energy

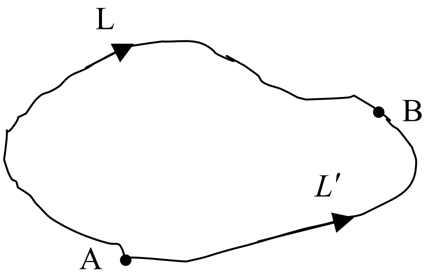
Gravitational force is a vector quantity, directed along the line **r** joining M and m.

$$F = \frac{GMm}{r^2}$$

$$\mathbf{F} = \frac{GMm}{r^2}\hat{r} = \frac{GMm}{r^3}\mathbf{r}$$

It is an *attractive*, *central*, *conservative* force.

A conservative force is one under which the work done in moving an object from A to B does not depend on the path taken between A and B.



$$W_{AB} = \int_A^B \mathbf{F.dr}$$

$$= W_L = \int_L \mathbf{F.dr}$$

$$= W_{L'} = \int_{L'} \mathbf{F.dr}$$

And

$$W_L - W_{L'} = 0 = \oint \mathbf{F.dr}$$

Or, equivalently, a conservative force is one under which the work done in moving an object around a closed loop is zero.

For a small displacement $d\mathbf{r}$, the change in potential energy of the body is $dU = -dW$, so that

$$dU = -\mathbf{F}.d\mathbf{r}$$

$$\mathbf{F} = -\frac{dU}{d\mathbf{r}} = -\frac{dU}{dr}\hat{\mathbf{r}} = -\frac{dU}{rdr}\mathbf{r}$$

We therefore have the *gravitational potential energy:*

$$U = -\frac{GMm}{r}$$

Total Energy of Gravitational System

Consider two point masses m and m', moving at velocities **v** and $\mathbf{v}'$ under their mutual gravitation, the total mechanical energy is

$$E = KE + PE = \tfrac{1}{2}mv^2 + \tfrac{1}{2}m'v'^2 - \frac{Gmm'}{r}$$

Let $m' >> m$, then if we use the Centre-of-Mass frame of reference we can ignore the kinetic energy of m'.

Now consider a circular orbit of m (a planet) around m' (the Sun). We have centripetal force equal to gravitational force,

$$\frac{mv^2}{r} = \frac{Gmm'}{r^2}$$

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}\frac{Gmm'}{r}$$

$$E = KE + PE = -\tfrac{1}{2}\frac{Gmm'}{r}$$

It is no accident that the total energy is negative. This distinguishes *closed* orbits (the planet is "captured" by the Sun's gravitational field), circular or elliptical, from *open* orbits. Open orbits are parabolic for $E = 0$ and hyperbolic for $E > 0$.

Escape Velocity: For a body to escape the gravitational field of another (e.g. a comet to leave the Solar System, or a rocket to escape the Earth ballistically), it must have sufficient speed so that $E = KE + PE$ is zero or positive. The escape velocity is defined by $E = 0$:

$$E = \tfrac{1}{2}mv_{esc}^2 - \frac{Gmm'}{r} = 0$$

$$v_{esc} = \sqrt{\frac{2Gm'}{r}}$$

Putting in the values for Earth's mass and radius, the escape velocity for a body at the Earth's surface is approx 1.13×10^4 m s^{-1}. Note that it does not matter what the *direction* of the velocity is (though it would be as well if it were not to result in a collision) – the direction merely determines which open orbit the body follows.

Orbital Speed: For a satellite in a circular orbit,

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}\frac{GMm}{r}$$

$$v = \sqrt{\frac{GM}{r}}$$

We cannot choose orbital speed and radius separately; for a given radius there is a given speed.

Orbital Period: The circumference of the orbit is $2\pi r$, so the period T is

$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{GM}} = \frac{2\pi r^{3/2}}{\sqrt{GM}}$$

in agreement with Kepler's 3rd Law and its derivations above.

Geosynchronous Orbits: To obtain an orbital period around the Earth of one day (useful for telecommunications), we solve

$$\frac{2\pi r^{3/2}}{\sqrt{GM}} = 24 \times 60 \times 60$$

$$r = 4.22 \times 10^{7}\ \text{m} = 42227\ \text{km} = 26239\ \text{miles}$$

This corresponds to an altitude of 42227 – 6380 km ~ 36000 km.

First proposed by Arthur C. Clarke in an article in Wireless World in 1945, there are now hundreds of geosynchronous satellites in a ring around the Equator. Google "J Track" and follow the link to J Track 3D to see them.

Time delay: At c, time to geosynchronous satellite and back is

$$\sim 7 \times 10^{7}\ \text{m} / 3 \times 10^{8}\ \text{m s}^{-1} \sim 0.25\ \text{s}$$

This is very noticeable on "live" satellite links on TV and in phone calls routed by satellite. For this reason, optical fibre links are preferred.