SPECIAL RELATIVITY

Albert EINSTEIN introduced his **SPECIAL THEORY OF RELATIVITY** in 1905. To understand the theory, we will first review the background, the theoretical and experimental developments since Newton.

SPECIAL means just, "not general". The theory is about inertial frames only. Recall that, for Galileo and Newton, such frames had a central importance in mechanics and cosmology, as, by the **PRINCIPLE OF RELATIVITY**, no laboratory-based measurement could establish one rather than another as "at rest", or "in motion".

Galileo and Newton were familiar with the relationships among inertial frames. If in the frame S the coordinates of an event are (x,t), then, in a frame S' moving at relative velocity v in the x-direction its coordinates are (x',t'). If we let the origins coincide at $x = x' = 0, t = t' = 0$, then the coordinates are related by

$$x' = x - vt$$

$$t' = t$$

These equations are the Galilean Transform. (Adding equations for y and z is trivial).

All of mechanics is *invariant* under the Galilean transform. This means that the same formulae for, e.g., the collisions of billiard balls or for the Coriolis force, are recovered if we substitute x' for x, t' for t, and then simplify. The velocity v simply drops out. Mechanics is the same in any inertial laboratory, whatever velocity it may be travelling at.

James Clerk MAXWELL took the experimental findings of Ampère, Faraday, and Coulomb on electricity and magnetism, and synthesised them into a beautiful system of four inter-linked equations (see EMF, Semester B). His Treatise on Electricity and Magnetism was published in 1873. Light turned out to be an electromagnetic wave, with a velocity of $c \approx 3 \times 10^8$ m s^{-1}.

It soon became clear that Maxwell's equations are not invariant under the Galilean Transform. One interpretation was that there is a "stationary" rest frame, in which light has the velocity *c*. *This would be the "luminiferous ether"*, a postulated medium for light to 'wave' in. However, this interpretation did not agree with experiment (stellar aberration, the Michelson-Morley attempt to measure the speed of the earth through the ether, etc).

Other people sought ways to retain the Principle of Relativity. Consider the more general linear transform,

$$x' = ax + bt$$

$$t' = \alpha x + \beta t$$

with the coefficients a, b, α and β being functions of v. It turns out that the simple requirement that the back transform (x and t expressed in terms of x' and t') restricts the transform to the form

$$x' = ax - avt$$

$$t' = \frac{1-a^2}{av}x + at$$

with the single parameter a appearing in each of the four coefficients.

Note that the Galilean Transform is obtained by putting $a = 1$.

The Lorentz Transform: It was Lorentz who noticed that Maxwell's Equations are invariant under this transform if a has the value

$$a = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

The transform with this value of a is therefore called the *Lorentz Transform*. Note that at ordinary speeds, a is very close to unity. E.g. for speeds typical of satellites, say 10 miles per second, $a = 1.00000000144$.

At first, no-one knew what this could mean. Fitzgerald noticed that this transform would shorten moving bodies, and this has led to the term, *Fitzgerald contraction*, being used in relativity.

The Lorentz transform applied to electric and magnetic fields does convert them into each other, so that an electric charge stationary in the frame S produces only electric field. Viewed from the frame S' moving relative to S, magnetic field is seen too. This is why Einstein's key 1905 paper was titled, "*On the Electrodynamics of Moving Bodies*" (in German).

What Einstein showed was that all of this required a new theory of space and time. We shall develop that theory, together with a rigorous definition of what it is that is being changed.

There are two major misconceptions in the literature:

- One is that Newton's Laws of Motion were shown to be untrue, or only approximately true at small velocities. That is not so. As outlined above, it is assuming that the Principle of Relativity is true, that Newton's Laws are true, and that Maxwell's equations are true, that forces a new theory of spacetime.
- The other is that Relativity derived from two axioms,
  - The Principle of Relativity
  - The Principle of the Constancy of the Speed of Light

In fact, Einstein often presented Relativity in this way in later books, and we will do so here.

SETTING UP A SPACE – TIME.

Events are fundamental. They happen in space and time – they have a where and a when. They are independent of a reference frame used to refer to them, but we have to set up a reference frame to give values to the *where* and *when* – to describe the events.

Galileo, Newton and others introduced the concept of an Absolute Frame of Reference which was defined to be at rest – Absolute Rest.

Then all frames moving at a constant velocity relative to this frame are called Inertial Frames of Reference. In these frames, Newton's Laws of Motion hold, without fictitious forces. Frames which are rotating or accelerating are called non-inertial and are dealt with in General Relativity.

Length and Time Measurement

Let frame S' be moving at constant velocity $\vec{v}$ relative to S. Let all axes be parallel, and $\vec{v}$ be along the *x*- axis of *S*. Take $t = t' = 0$ when the origins coincide, when $O = O'$.

Events are observed in both frames, and labelled by coordinates in space and time, (x, y, z, t) or (x', y', z', t'). The spatial coordinates are measured by metre rules or yardsticks and the temporal coordinate by clocks.

Comparison of observations in S and S' requires comparison of the yardsticks and clocks used by the observers in S and S'. It is easy to compare two yardsticks when they are stationary – just lay them alongside each other – and equally easy to synchronise two stationary clocks and to check that they run at the same rate. Then we make two assumptions based on the Principle of Relativity:

- Once yardsticks are compared with each other at rest, their lengths remain unchanged when in motion.
- Once clocks are synchronised and compared, they run at the same rate when in motion.

Note that these assumptions hold in Newtonian space-time; we shall see later that they both remain true in Special Relativity. (That is, there is no contraction of a moving yardstick of the sort Fitzgerald envisaged.)

Let event *P* occur at the position $\vec{r} = (x, y, z)$ at time *t* in S, and at the position $\vec{r}' = (x', y', z')$ at time t' in S'. Then

$$\vec{r} = \vec{v}t' + \vec{r}'$$
$$t = t'$$

is the Galilean transformation that gives coordinates of *P* in S if coordinates in S' are measured. And

$$\vec{r}' = -\vec{v}t' + \vec{r}'$$
$$t = t'$$

is the Galilean transformation that gives coordinates of *P* in S' if coordinates in S are measured.

Transform of Length

Let a rod of length L' be at rest in S'. Measurement of its length consists of finding the coordinates of its ends. This requires two events, which need not be simultaneous, yielding, e.g., x_1' and x_2' and a length of $L' = x_2' - x_1'$. An observer in S, however, will want the two measurements events to be simultaneous. The length transforms as

$$\begin{aligned} L' = x_2' - x_1' &= -vt_2 + x_2 + vt_1 - x_1 \\ &= (x_2 - x_1) - v(t_2 - t_1) \end{aligned}$$

which gives the right answer, $L = x_2 - x_1$, only if $t_2 = t_1$ (or if the correction for lack of simultaneity is made).

Measurements of the same physical quantity, made in S and S', are made under different physical conditions.

Transform of Time Intervals

Let a clock be at rest in S' and two events P_1 and P_2 take place next to it (i.e. at the same place.) The observer in S' need only look at the clock to observe the positions of its hands at the two events.

The observer in S sees the two events occurring at two different places, and so he has to look at two separate clocks in his own frame (or at one moving clock in the S' frame).

Measurements of the same physical quantity, made in S and S', are made under different physical conditions.

Transforms of Velocity and Acceleration

We merely need to differentiate with respect to t or t'

$$x' = x - vt$$

$$u' = \frac{dx'}{dt'} = \frac{dx}{dt'} - v\frac{dt}{dt'} = \frac{dx}{dt} - v = u - v$$

$$a' = \frac{d^2x'}{dt'^2} = \frac{d^2x}{dt^2} - \frac{dv}{dt} = a$$

In general,
$\vec{u}' = \vec{u}' - \vec{v}'$
$\vec{a}' = \vec{a}$

Transforms of Mass and Force

Observers in S and S' agree on acceleration. They achieve consistency if, having written,

$$F = ma$$

$$F' = m'a'$$

they agree that mass and force are unchanged and that Newton's 2^{nd} Law is unchanged. (*That is what invariant under a transformation means.*)

Transforms of Momentum and Energy

It follows that observers in S and S' agree that momentum and mechanical energy are conserved, even though they will disagree on the values (as they depend on $\vec{v}$).

Consider a ball thrown with initial velocity components upwards, v_{0y} and horizontal, v_{0x} in S. The observer in S will write,

$$½ E_{Tot} = mgh + ½mv_{0x}^2 + ½mv_y^2 = constant \text{ in } S$$

The observer in S' moving at v_{0x} will write

$$E'_{Tot} = m'g'h' + ½m'v'^2_y$$

Since, as we have seen,

$$m' = m$$

$$g' = g$$

$$h' = h$$

$$v'_y = v_y$$

it follows that E'_{Tot} differs only by a constant from E_{Tot} and is therefore also conserved.

The analysis for momentum is very similar, and leads in the same way to the conclusion that p' differs from p only by a constant and is therefore also conserved.

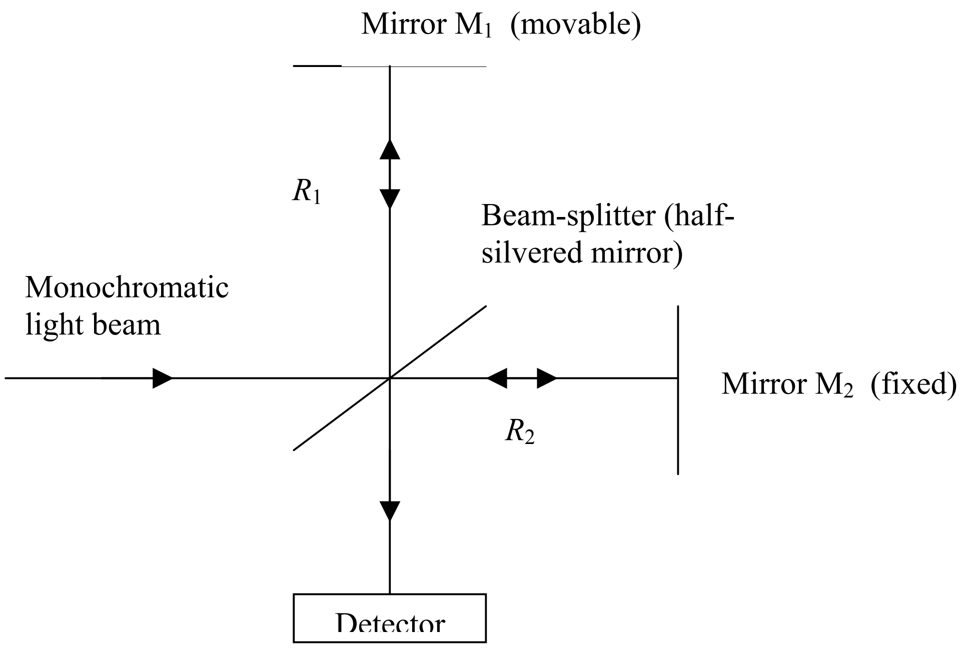
Transforms in Special Relativity

It is most elegant to proceed directly from the Lorentz invariance of Maxwell's equations of electromagnetism. However, this is mathematically rather difficult. Most books, and indeed Einstein's own later books, proceed from a *deduction* from Maxwell's equations – that the speed of light is $c \sim 3 \times 10^8$ m s^{-1}. Invariance of physical laws to the velocity of the laboratory requires that the speed of light is then c in any inertial frame. If two observers, in S and in S', measure the velocity of the *same* light beam, they must both obtain the value c. This contradicts the Galilean transform, according to which if one observer measures c, the other must obtain $c - v$.

There was some experimental evidence for a constant speed of light. First, from observations of the moons of Saturn, it was clear that the speed of light does not depend on the velocity of the source – unlike a rifle bullet. Since light was known to be a wave, this was not surprising, since waves travel at a speed defined relative to the medium, not the source. A medium for light waves called the *Aether* had been postulated. Michelson and Morley (1887) proposed to measure the speed of the Earth through the aether, by comparing the speed of light parallel to the Earth's motion and perpendicular. They failed. This was the second piece of experimental evidence for a constant speed of light.

The Michelson-Morley Experiment.

The experiment uses a Michelson Interferometer:



The two arms are of lengths R_1 and R_2. If $R_1 = R_2$, the two paths interfere *constructively* and the detector detects light. This is called a *bright fringe*. If the path lengths are different, then if

$$R_1 - R_2 = \frac{n\lambda}{2}$$

interference is still constructive and a bright fringe is observed. If

$$R_1 - R_2 = \frac{n + ½}{2}\lambda$$

then interference is *destructive* and no light falls on the detector (a dark fringe). If M_1 is moved, a succession of light and dark fringes is observed.

Use as a Speedometer for the Earth

The idea is to put the instrument with one arm (say, arm 1) parallel to the Earth's motion, and the other arm, 2, perpendicular to the motion. The velocity of the Earth in its orbit around the sun is $v \sim 10^{-4}\, c$, so this is the kind of result expected.

If the light takes time $½t_2$ to go from the beam-splitter to mirror M_2, the apparatus moves a distance $s = ½t_2c$ in that time. The distance the light travels is not R_2, but

$$l_2 = \sqrt{R_2^2 + \left(\frac{vt_2}{2}\right)^2}$$

So

$$t_2 = \frac{2l_2}{c}$$

$$t_2^2 = \frac{4}{c^2}\left(R_2^2 + \left(\frac{vt_2}{2}\right)^2\right) = \frac{4R_2^2}{c^2} + \frac{v^2t_2^2}{c^2}$$

$$t_2 = \frac{2R_2}{c}\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

In beam 1, the time taken to go from the beam-splitter to the mirror and back is

$$t_1 = \frac{R_1}{c - v} + \frac{R_1}{c + v} = \frac{2R_1}{c}\frac{1}{1 - \frac{v^2}{c^2}}$$

So the *difference in time* in arms 1 and 2 is

$$\Delta t = t_2 - t_1 = \frac{2}{c}\left(\frac{R_2}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{R_1}{1 - \frac{v^2}{c^2}}\right)$$

Letting $R_1 = R_1 = R$, putting $\beta = v/c$ and using the binomial theorem (v^2/c^2 is small),

$$\Delta t = -\frac{R\beta^2}{c}$$

This translates into an apparent pathlength difference of $c\Delta t = R\beta^2 \sim 10^{-8} R$.

This pathlength difference may be expressed as a number of wavelengths,

$$n = \frac{c\Delta t}{\lambda} = \frac{R\beta^2}{\lambda}$$

which for R = 22m (using a multiple path instrument), $\beta = 10^{-4}$, λ = 550nm, gives n = 0.4.

The apparatus is now rotated through 90°, interchanging the two arms. The fringe shift is

$$\Delta n = 2n$$

Michelson and Morley reckoned they could see a fringe shift as small as n = 0.01.

Their measured result was null, i.e. $n = 0 \pm 0.01$.
(Later improved versions of the experiment achieved $n = 0 \pm 0.002$.)

Conclusion: The apparatus does not work as a speedometer for the Earth.

One interpretation is that the speed of light is measured to be c whatever the speed of the laboratory. That Electromagnetism has no absolute frame of reference. So this result has very often been used (by Einstein himself and by most textbook writers to justify stating two **Postulates** and deriving Special Relativity from them:

- **The Principle of Relativity**
- **The Principle of the Constancy of the Speed of Light.**

We shall follow that path, despite Einstein's own claim that he hadn't heard of the Michelson-Morley experiment by 1905 when he published Special Relativity!