7. GEOMETRIC OPTICS

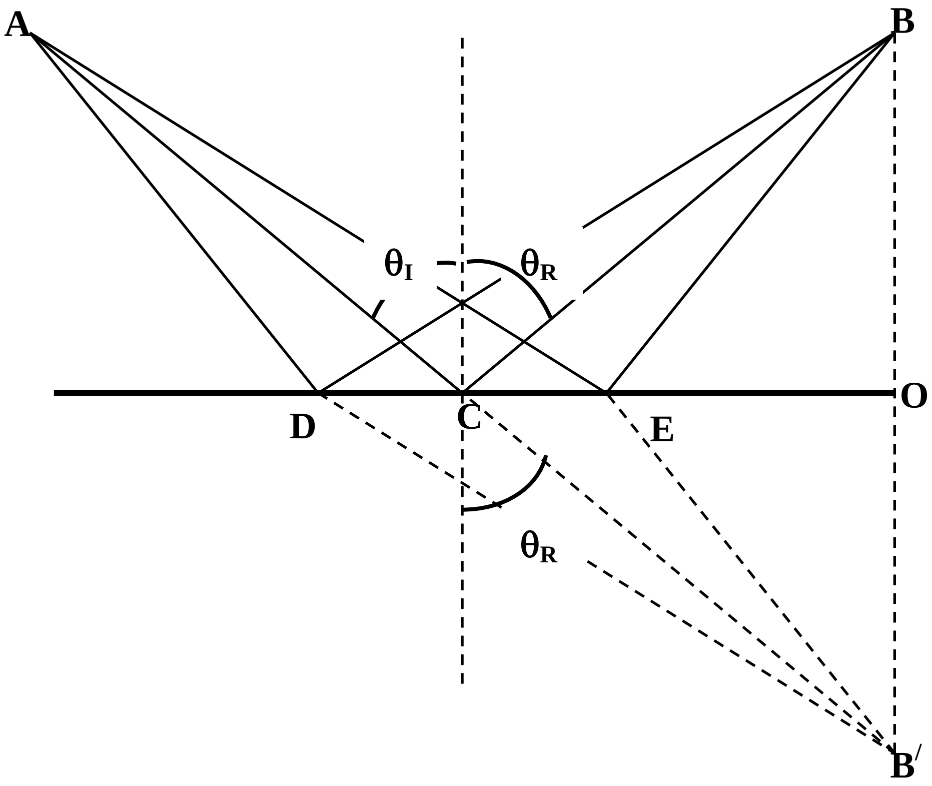
Introduction.

Previously the implications of the wave nature of electromagnetic radiation including amplitude and phase have been studied. Notably the effects of polarisation, interference and diffraction were examined using the principle of superposition of electric fields. One of the first things we examined however was how the boundary conditions on the electric and magnetic fields controlled what happened to these electromagnetic waves at the interface between two dielectrics, ie. two semi infinite media with differing polarisabilities and consequently differing refractive indices. There, it was discovered that the radiation could either reflect back into the sourced region from the boundary or refract as it is transmitted into the unsourced medium. Optical systems, arrangements of elements that act on light to produce specific effects translating an object into an image at a separate location can be analysed to a great extent by ignoring the wave nature of light and dealing with rays. ***The ray represents the direction of travel of the energy in a light wave and this direction is normal to the wavefront that we have previously dealt with.*** The study of light using rays as the basic elements through which the behaviour of light is understood is known as **<u>geometric optics</u>** and can offer great simplification over the **<u>physical optics</u>** approachwhere the emphasis was on the wave nature of light. ***It can only be used with confidence when the dimensions of the elements making up the physical system are much larger than the wavelength of the light involved.*** In fact it is the case that

$$\underset{\lambda\to 0}{\text{Limit}}\left(\text{Physical Optics}\right)=\left(\text{Geometric Optics}\right)$$

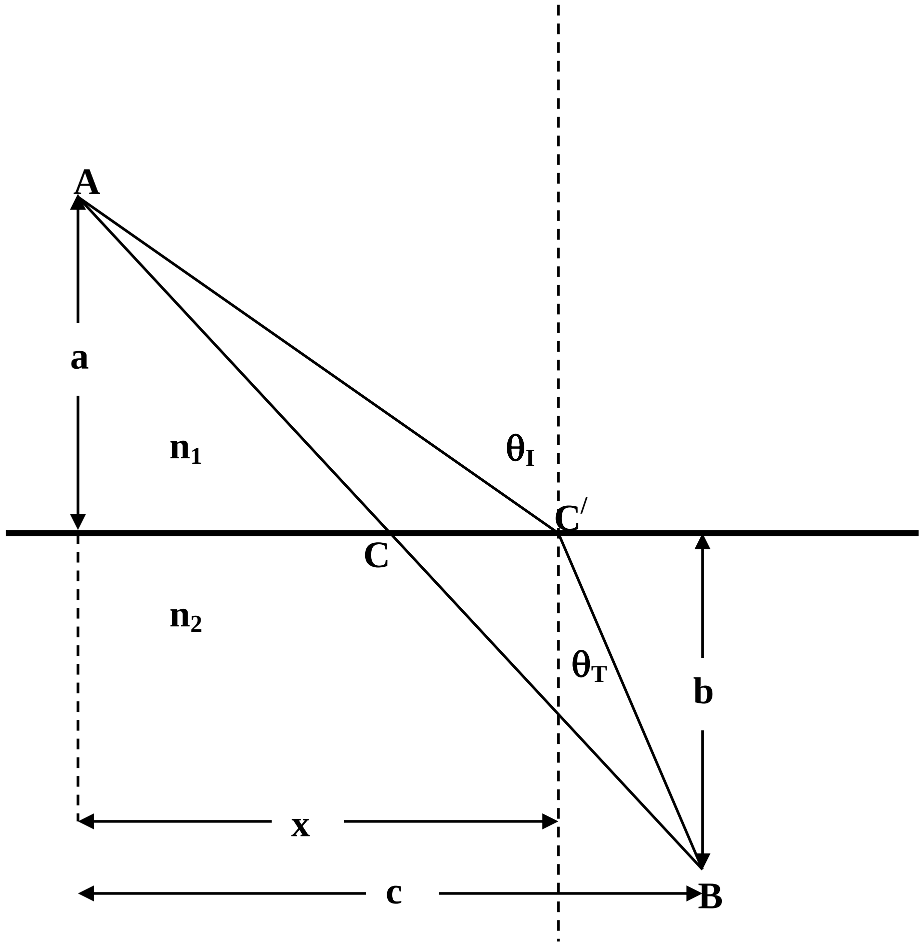
Huygens principle, where each point on a wavefront acted as a secondary source of spherical waves whose amplitude was the only important thing to be considered, embodied a method of calculating the actions of optical systems that we saw had limitations when it came to dealing with the effects of diffraction and that those effects required a modification to become the Huygens-Fresnel principle, where the phases of the secondary spherical waves were to be included. Going from the former to the latter was effectively crossing the line from a geometric approach to a physical approach. A further important and simplifying principle in the geometric optics approach is **<u>Fermat's</u>**

principle of least action. Fermat's principle follows on from a hypothesis put forward by Hero of Alexandria who lived in the second century BC. He proposed that when light travelled between two points it took the shortest possible distance. In a homogeneous medium this would obviously be a straight line. Hero's law will easily give the law of reflection. This is easily seen in the following construction.



Three potential paths are shown for a ray of light to get from A to B via reflection in some plane. We also construct the perpendicular from B through the plane at O to an equal distance below the plane such that OB = $OB^/$ Each of the three reflected rays, DB, CB and EB is shown reflected below the plane travelling to $B^/$. It is clear (assuming Euclidean geometry) that the shortest path from A to $B^/$ is the straight line $ACB^/$ and it is also clear that BC = $B^/C$ as the triangles OBC and $OB^/C$ are similar thus the straight line $ACB^/$ is the equivalent to the path ACB where $\theta_I = \theta_R$. The law of reflection is therefore proved using Hero's principle.

Hero's principle, however, fails to deal with refraction as we see in the next construction



The correct line between the two points A and B when each lies in a medium of differing refractive index is $AC^/B$ ($n_2 > n_1$) and not the straight line ACB and Hero's principle fails. **Pierre de Fermat** tried to rescue the principle by demanding not that the shortest distance between two points be the defining feature of the path travelled but that **the path that takes the least time to travel defines the preferred path**. The ray will travel more slowly in the medium with the higher refractive index and will therefore need to travel a shorter distance in that medium in order to minimise the time of the total journey

$$\tau = \frac{AC}{v_1} + \frac{CB}{v_2} = \frac{n_1}{c}\sqrt{x^2 + a^2} + \frac{n_2}{c}\sqrt{b^2 + (c - x)^2} \tag{7.1}$$

To minimise the time taken we need to make sure that $\frac{d\tau}{dx} = 0$

$$\frac{d\tau}{dx} = \frac{n_1}{c}\frac{x}{\sqrt{x^2 + a^2}} - \frac{n_2}{c}\frac{(c - x)}{\sqrt{b^2 + (c - x)^2}} = 0 \tag{7.2}$$

$$n_1 \frac{x}{\sqrt{x^2 + a^2}} = n_2 \frac{(c - x)}{\sqrt{b^2 + (c - x)^2}} \qquad (7.3)$$

The terms under the square roots are AC and BC the hypotenuses of triangles $AOC^{/}$ and $C^{/}BO^{/}$ and therefore applying Fermat's principle of least action gives

$$n_1 \sin\theta_I = n_2 \sin\theta_T$$

which is Snell's law of refraction.

The use of a ray optic approach and Fermat's principle has allowed us to establish the two laws governing the behaviour of light at interfaces between dielectric media, two laws that were previously found with the use of Maxwell's equations and their boundary conditions at the interface via a physical optic approach. Geometric optics sets out to establish what happens where such boundaries exist in the more complicated circumstances that may arise when these boundaries are designed and shaped to perform complex imaging functions.

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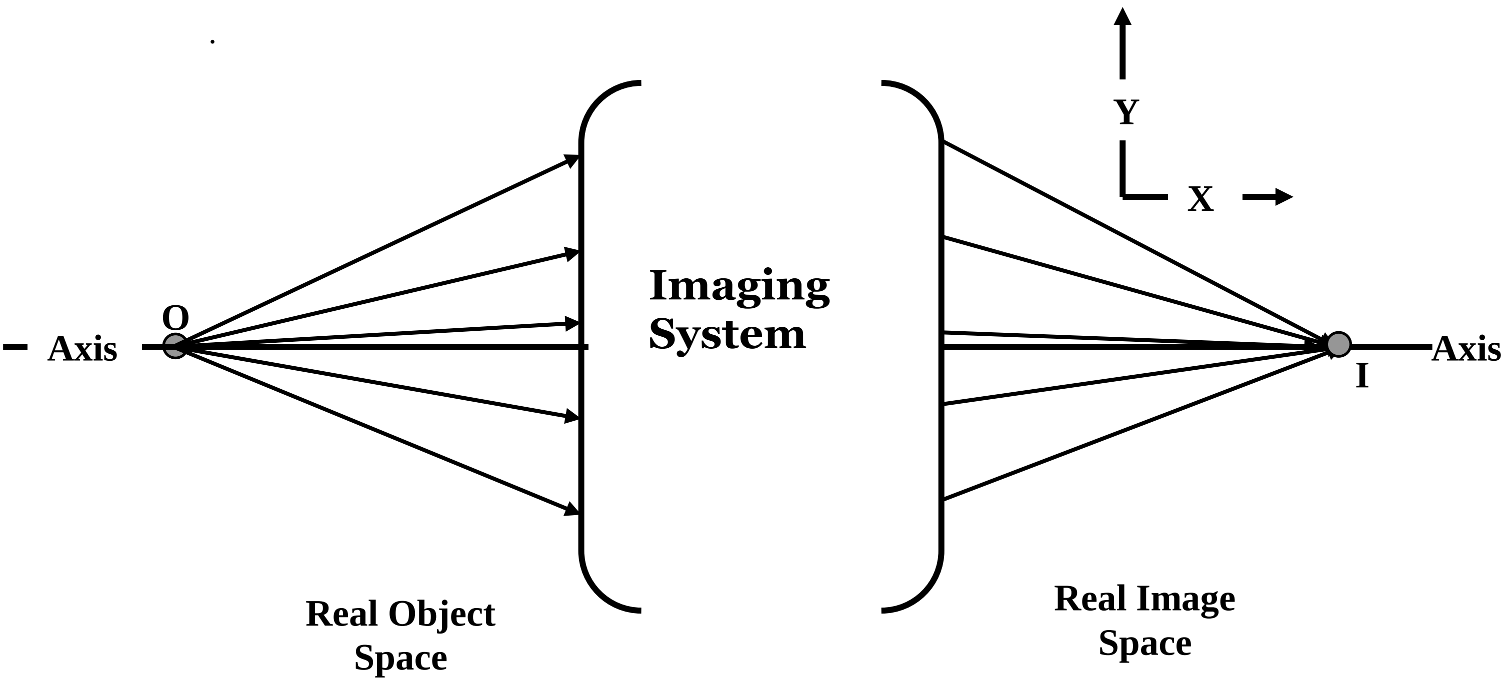


We begin by making some general onservations on imaging systems before looking at specific examples of imaging systems.

We begin by talking about an abstract imaging system composed of any number of refracting and reflecting surfaces that acts upon rays to redirect them. The generalised system is represented in the simple figure above.

We assume the rays to be travelling from left to right in what follows. On the left is what is known as the real object space, and every point, O, in object space is related, via our unspecified imaging system, to a point I in the real image space where due to the action of the system the rays from O converge. All of the rays drawn are taking the minimum time to reach I from O according to Fermats principle ie. they are isochronous. Also from the principle of reversibility the diagram may be reversed with the rays travelling from I to O. In an ideal system every ray from the object point, O, that passes through the imaging system, and only these rays, will pass through the image point I. O and I are said to be ***conjugate object and image points***. Other points in the object space will have their own conjugate image point in the image space obeying the same rules. When the object is at infinity in object space the conjugate point in image space is on the axis at a focal point F_i and conversely when the image is at infinity in the image space the object must be at the focal point F_o in object space. The two spaces are said to be linked by a mapping described by a projective transformation that contains a linear relationship between axial distances, X, either side of the optical system

$$\frac{f_o}{X_o} = \frac{X_i}{f_i} = -\frac{Y_i}{Y_o}$$

as well as transverse distances (perpendicular distances from the axis) , Y, where the distances are measured from the focal points F in the object or image space or from the axis respectively.

In practice, there will be some departure from ideality due to:

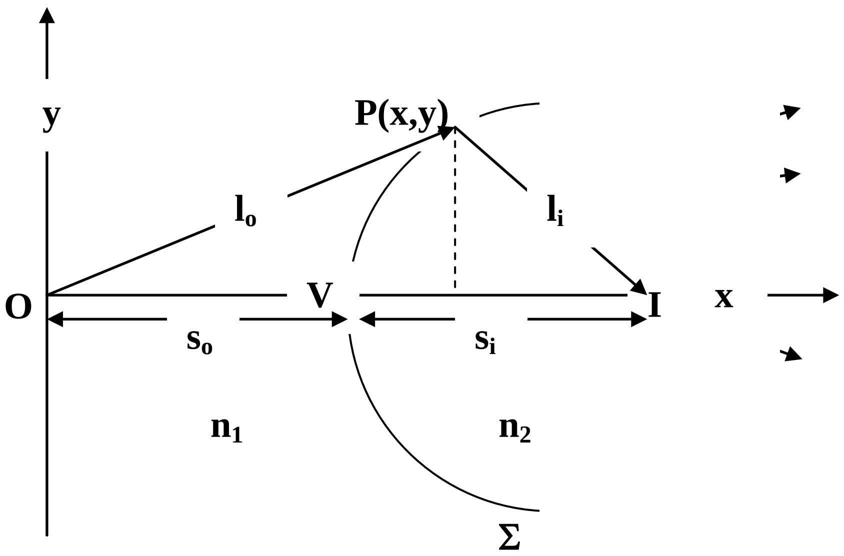
(i) **Scattering.** For example there will be reflection of some intensity at dielectric interfaces within the imaging system and there will also be some scattering within media where there is departure from homogeneity, eg. in glass there may be small local density fluctuations leading to refractive index variations acting as scattering centres. These effects lead to a less bright image.

(ii) **Aberrations.** Spherical, astigmatic and chromatic aberations will be present where the components of the system depart from the ideal form or where different wavelengths/colours are refracted differently. These will, when uncorrected lead to non-ideal performance.

(iii) **Diffraction.** The wave nature of the light will have an effect at some level leading to a blurring of the image point.

These will be discussed later. For now we deal with perfection!

Surfaces that reflect or refract perfectly are called Cartesian surfaces. For a Cartesian surface to produce a perfect image by refraction, is not a staightforward matter.



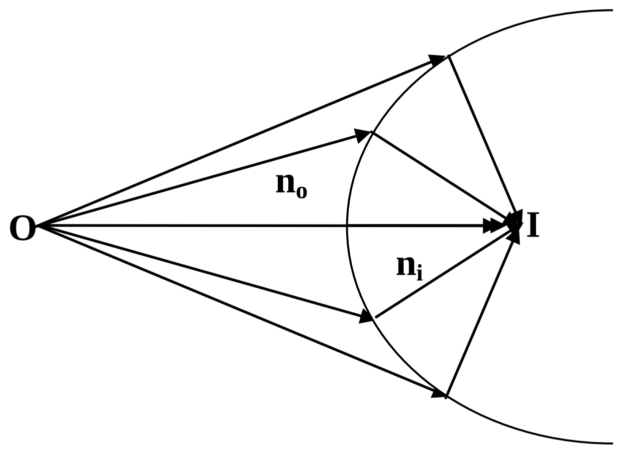
Consider the refraction occuring in the above diagram. There is an object at O, a refracting surface boundary, Σ , separating media of refractive index n_1 and n_2 and an imaged formed at I. There are the rays passing towards and from P, an arbitrary point on the surface and the ray passing directly from O to I via the vertex at V. Fermats principle needs to be upheld with both routes OPI and OVI taking the same time to be traversed. Fermats principle can be stated in terms of the optical path lengths which must be equal. If a ray travels a distance l in a medium of refractive index n then the optical path length is nl and this must be the same for any path by which a ray travels from one point to another. In the present case

$$n_1 l_o + n_2 l_i = n_1 s_o + n_2 s_i = constant \quad (7.4)$$

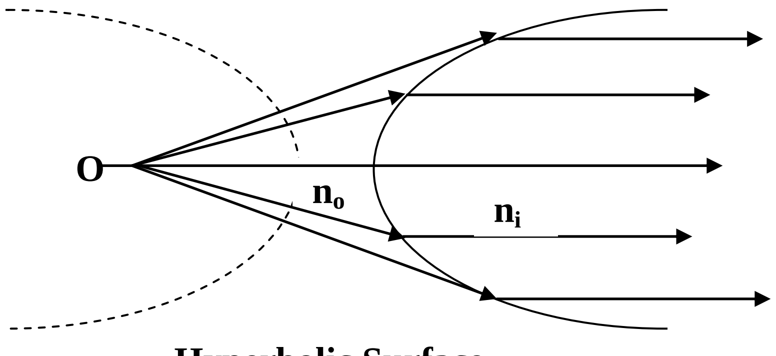
From the diagram

$$n_1\sqrt{x^2 + y^2} + n_2\sqrt{y^2 + (s_0 + s_i - x)^2} = constant \qquad (7.5)$$

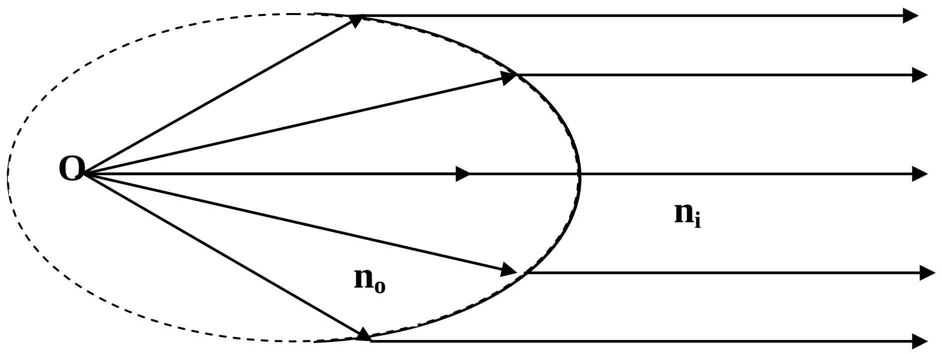
This equation describes a Cartesian ovoid of revolution which will be better specified once the problem under consideration is more precisely defined. Usually we require the object and image to be in the same medium (air) and this requires a minimum of two refracting surfaces.



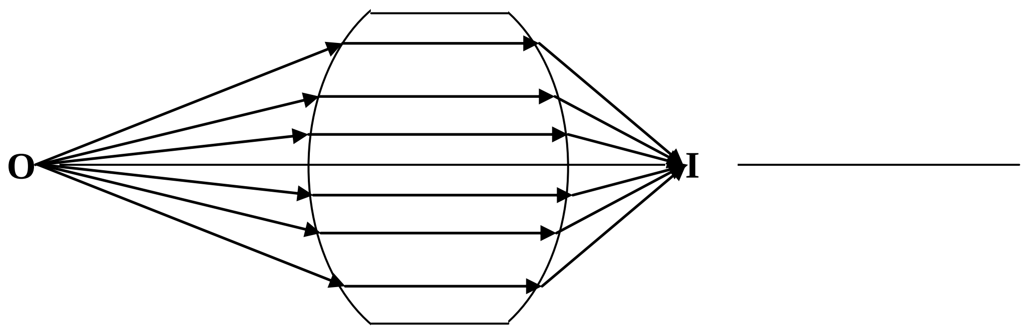
Cartesian Ovoid



Hyperbolic Surface



Ellipsoid Surface



Double Hyperbolic Lens

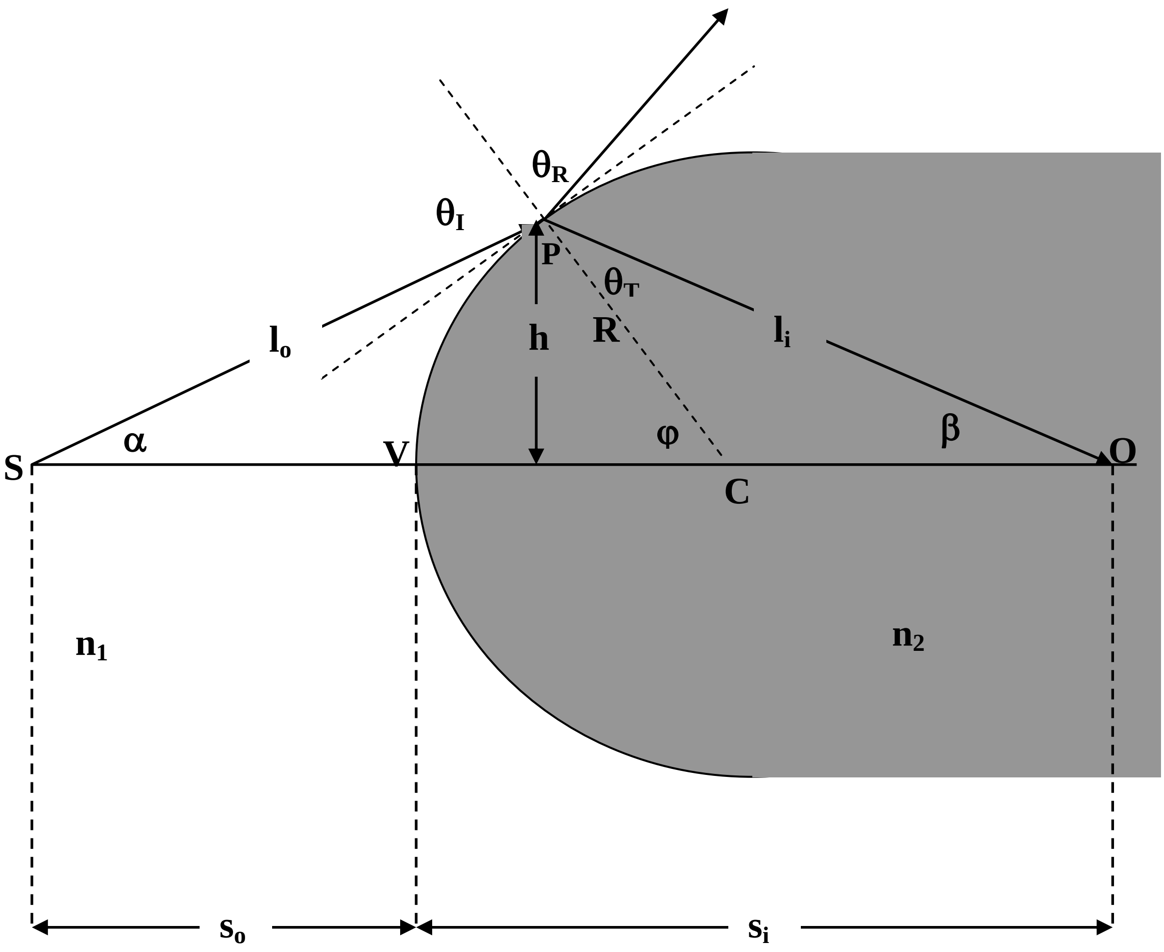
The easiest example of of a lens that will provide a perfect image through refraction is the double hyperbolic lens whose entrance surface will refract the rays into parallel rays before the second surface refracts the series of parallel rays to a common image point.

Any lens is an example of a dielctric interface that has been shaped in order to achieve some desired effect eg the interface between a curved glass surface and air/vacuum. The easiest shape to create (by precision grinding) is a section of a sphere although this will not provide a perfect object/image system it will not depart too far from ideality and we will deal with so called spherical aberations later.

Refractive Optics.

Refraction at spherical surfaces.

The diagram below illustrates an ideal spherical surface of radius R centred at C the sphere having a refractive index n_2. It is in a medium of refractive index n_1. There is a point source S. A ray from S to a point, P at the interface is shown at an angle of incidence θ_I to the **local normal** at P. Also shown are reflected and transmitted rays obeying the usual laws of reflection and refraction. Because $n_2 > n_1$ the transmitted ray is refracted towards the normal and thus towards the optical axis of the system OS crossing the axis at the point O. If we imagine the 3D diagram that is the solid of revolution created by rotating the figure about the optical axis then all rays travelling from the source at the same angle to the optical axis (ie. in a cone) will cross the optical axis at O.



In the triangle CPS we see $\alpha + \varphi = \theta_I$ and in triangle CPO we see $\varphi = \theta_T + \beta$.
From Snell's law (of refraction)

$$n_1 \sin\theta_I = n_1 \sin(\alpha+\varphi) = n_2 \sin\theta_T = n_2 \sin(\varphi-\beta) \qquad (7.6)$$

Using $\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$

$$n_1[\sin\alpha\cos\varphi + \cos\alpha\sin\varphi] = n_2[\sin\varphi\cos\beta - \cos\varphi\sin\beta] \qquad (7.7)$$

We now use an important approximation called ***the paraxial approximation*** where the rays are assumed to be travelling close to the axis **making α, β and φ small** and **approximately equal to their sin** and **their cosine being approximately 1**.
(Eg. 20^0 = 0.348 rads and sin0.348 = 0.342)
Alternatively stated the paraxial approximation assumes h is much smaller that R and l

$$n_1\left[\frac{h}{l_0} + \frac{h}{R}\right] = n_2\left[\frac{h}{R} - \frac{h}{l_1}\right] \qquad (7.8)$$

Putting $l_0 \approx s_0$ and $l_1 \approx s_1$

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} \qquad (7.9)$$

This result could have been obtained in a different fashion by using Fermat's principle which requires that the optical path length is stationary wrt position

$$\Lambda = n_1 l_o + n_2 l_i \qquad (7.10)$$

With use of the cosine rule in triangles SPC and PCO to get expressions for l_o and l_i respectively,

$$l_o = \left[R^2 + (s_o + R)^2 - 2R(s_o + R)\cos\varphi\right]^{1/2} \qquad (7.11a)$$

And

$$l_i = \left[R^2 + (s_i - R)^2 + 2R(s_i - R)\cos\varphi\right]^{1/2} \tag{7.11b}$$

the optical path length can be rewritten

$$\Lambda = n_1\left[R^2 + (s_o + R)^2 - 2R(s_o + R)\cos\varphi\right]^{1/2} + n_2\left[R^2 + (s_i - R)^2 + 2R(s_i - R)\cos\varphi\right]^{1/2} \tag{7.12}$$

There is a very important sign convention to be observed when doing calculations in geometric optics and it is introduced here wrt the above diagram. The convention as far as we have used it so far is for light propagating left to right (ie. object space to the left and image space to the right)

- **i)** **Distances relating to the object, eg s_o , are positive when measured to the left of V** whilst
- **ii)** **Distances relating to the image, eg. s_i , are measured positive when measured to the right of V**.
- **iii)** **R is measured as positive if the centre of the circle is to the right of V**.
- **iv)** **The object or image height, h, is measured positive when measured above the optical axis.**

All of the quantities in that diagram are measured as positive under this convention.

$$\frac{d\Lambda}{d\varphi} = \frac{n_1 R(s_o + R)\sin\varphi}{2l_0} - \frac{n_2 R(s_i - R)\sin\varphi}{2l_i} = 0 \tag{7.13}$$

$$\frac{n_1(s_o + R)}{l_0} = \frac{n_2(s_i - R)}{l_i} \tag{7.14}$$

Re-arranging

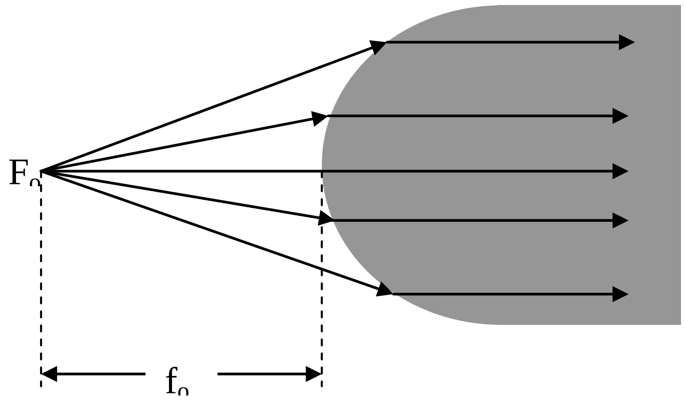
$$\frac{n_1 R}{l_o} + \frac{n_2 R}{l_1} = \frac{n_2 s_i}{l_i} - \frac{n_1 s_o}{l_o} \tag{7.15}$$

In the paraxial approximation $l_o \approx s_0$ and $l_i \approx s_i$

$$\frac{n_1 R}{l_o} + \frac{n_2 R}{l_1} = n_2 - n_1 \tag{7.16}$$

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} \tag{7.17}$$

Equation 7.17 is a very important equation in ray optics describing refraction at a single surface and the relation between the conjugate points and the radius of curvature of the surface. We will use 7.17 frequently and proceed to use it now to look at some special cases.



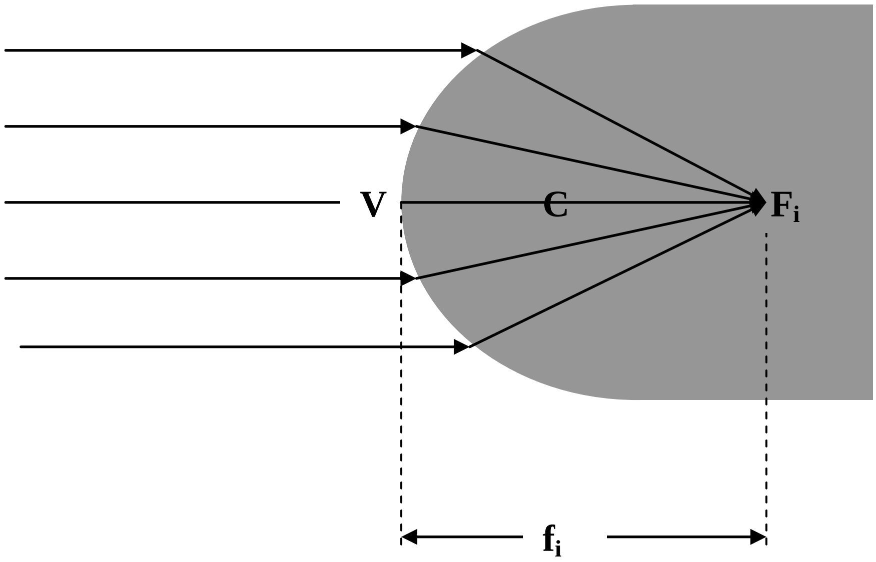
We may now use 7.17 to establish the conjugate points in some special circumstances.

1. When the object point source is placed at a point F_o a distance $f_o = s_o$ **from the vertex of the refracting surface** such that **the resulting image is formed at an infinite distance, $s_i = \infty$** the equation becomes

$$\frac{n_1}{f_o} + \frac{n_2}{\infty} = \frac{n_2 - n_1}{R} \tag{7.18}$$

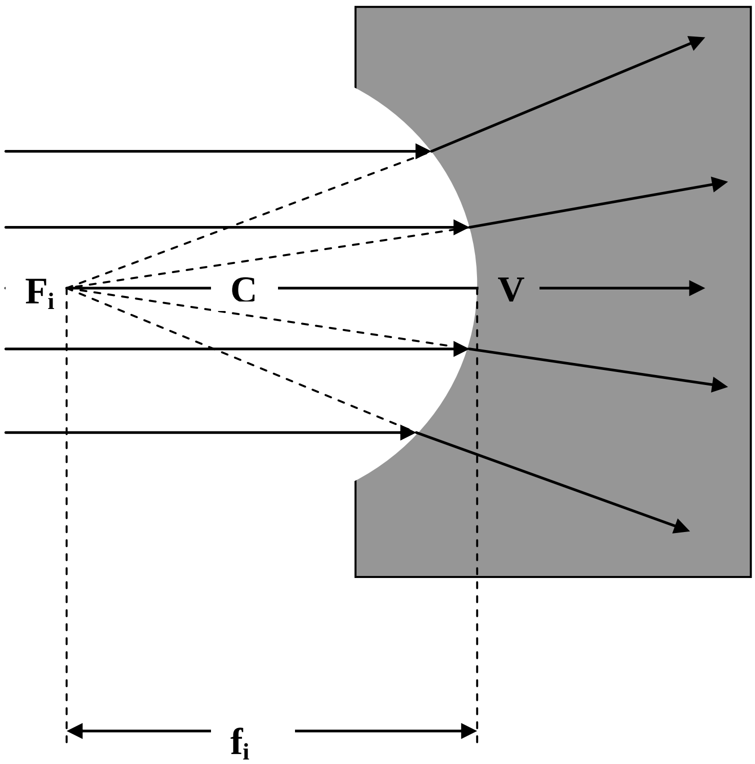
$$f_o = \frac{n_1}{n_2 - n_1} R \tag{7.19}$$

Where **F_o is the object (or first) focus** and **f_o the object (or first) focal length**.



The second focus is the point on the axis where the rays from an object at infinity, or alternatively plane waves are converted to converging spherical waves crossing the optical axis (or converging to a point) at a distance to the right of the vertex. Where this occurs is the image (or second) focal length, f_i .

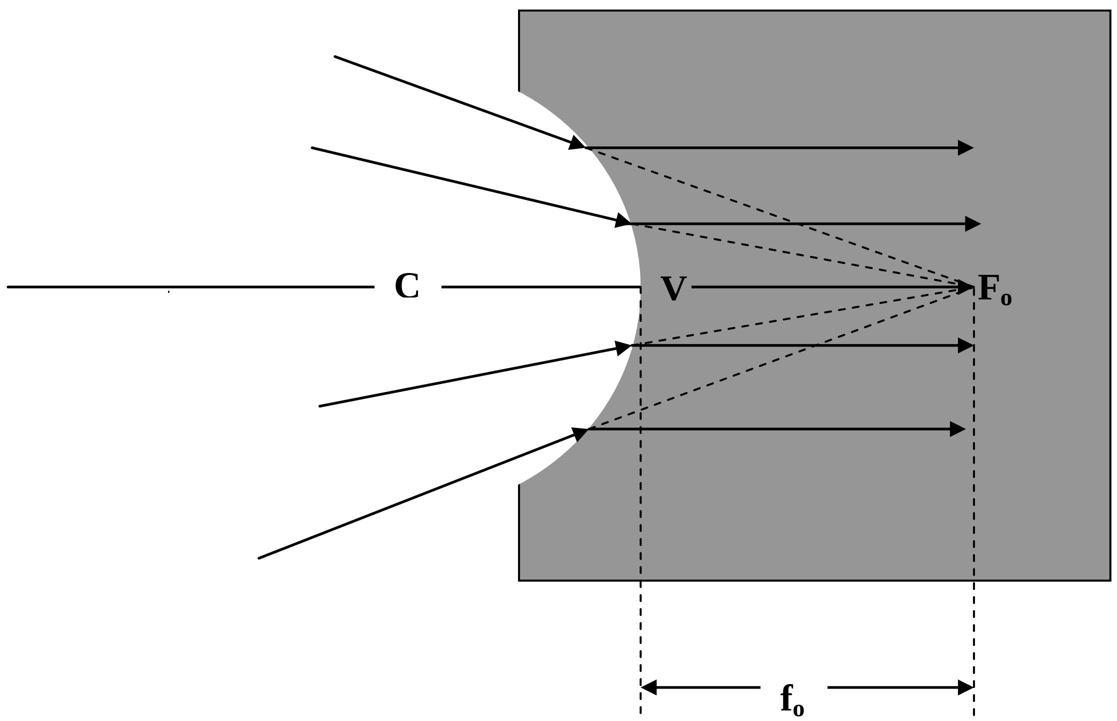
$$f_i = \frac{n_2}{n_2 - n_1} R \tag{7.20}$$



The above figure shows a spherical refracting surface where the high refractive index region is to the right and the centre of curvature is to the left of the vertex. Following the convention this means that R must be a negative number. We see from the construction that if the object is at infinity, $s_o = \infty$ and we have plane waves impinging on the surface the rays are refracted away from each other and appear to converge at a point , the image focus to the left of the vertex, **this makes the image focal length negative according to convention** and in fact **the image is a virtual image**, that is to say **it is not an image that could be projected onto a screen** as to form it we need to project the rays back. We can form the equation in this circumstance to find f_i

$$f_i = \frac{n_2}{n_2 - n_1} R = \frac{-n_2}{n_2 - n_1} |R| \qquad (7.21)$$

The fact that R is negative means that the equation gives a negative value for f_i as required.



Finally, the above figure shows the first focal point for the geometry where the centre is located to the left of the vertex and R is thus negative. Applying the equation

$$f_o = \frac{n_1}{n_2 - n_1} R = \frac{-n_1}{n_2 - n_1} |R| \qquad (7.22)$$

In this case the object is to the right of the vertex and the distance must be negative as is the case according to the equation. **This is a virtual object!** We will see later that **virtual objects do exist as the virtual image formed by another lens or refracting system**.

We now turn to the usual situation where the object and image are formed in the same medium, usually air. For this to happen we need at least **a pair of refracting surfaces or lens.**

With one hemispherical surface we have found;

$$\frac{n_1}{s_o}+\frac{n_2}{s_i}=\frac{n_2-n_1}{R}$$

Following the convention for the signs of distances;

The conjugate points for a positive curvature refractive surface are;

i) The object focal length (where the image is formed at $s_i = \infty$

$$f_o=\frac{n_1}{n_2-n_1}R \qquad \text{is positive}$$

ii) The image focal length (where the object is placed at $s_o = \infty$

$$f_i=\frac{n_2}{n_2-n_1}R \qquad \text{is positive}$$

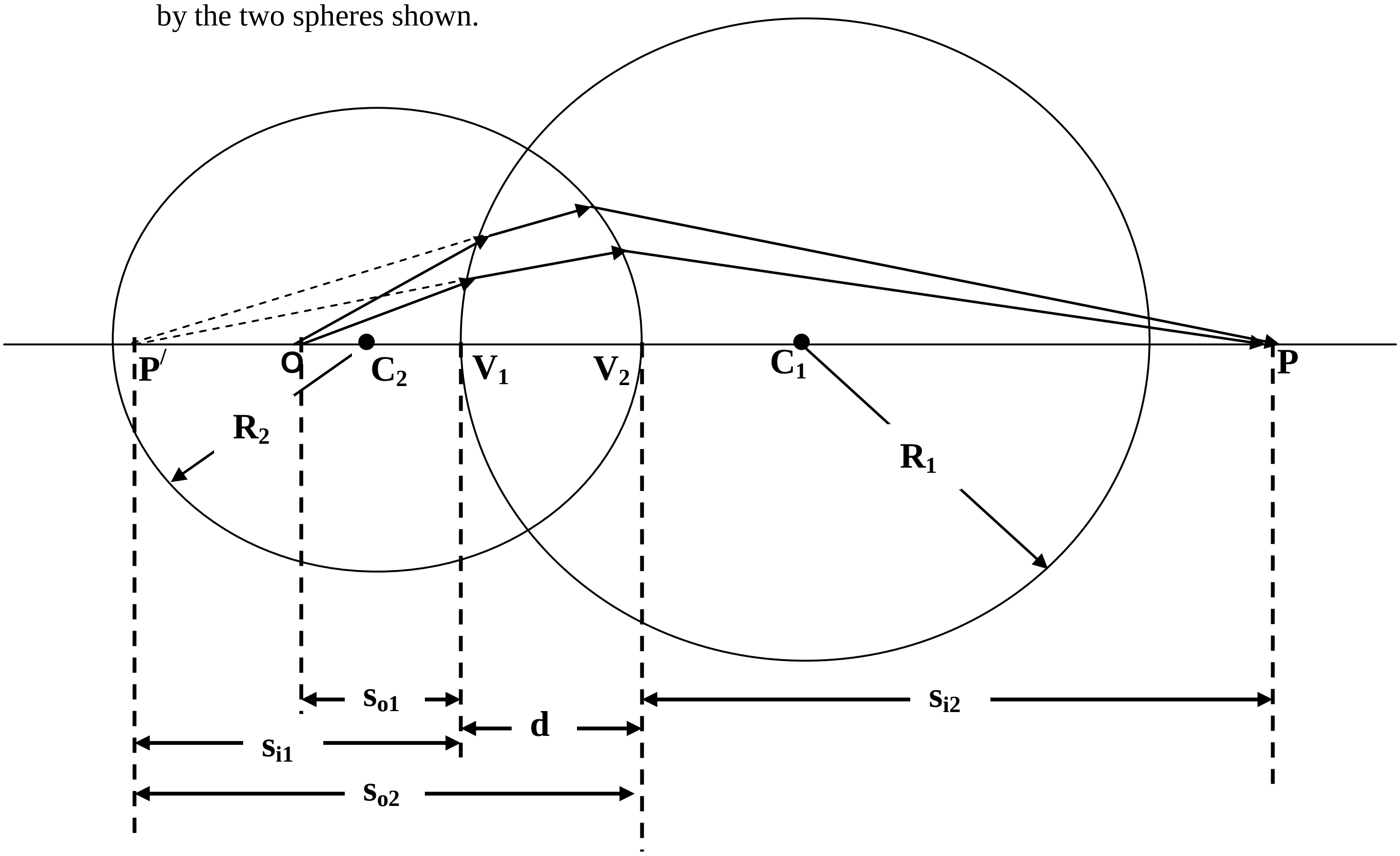
The corresponding conjugate points for a negative curvature refractive surface are;

$$f_o=\frac{n_1}{n_2-n_1}R=\frac{-n_1}{n_2-n_1}|R| \qquad \text{is negative}$$

$$f_i=\frac{n_2}{n_2-n_1}R=\frac{-n_2}{n_2-n_1}|R| \qquad \text{is negative}$$

c) Thin lenses.

The principles that we have applied to a single spherical refracting surface apply to two surfaces back to back allowing the determination of the property of a lens. With two spherical surfaces to combine there are a number of general possibilities that can be identified before specific combinations can be looked at. First we examine the simple proposition of a bi convex lens as represented in the following diagram. First it is important to locate the conjugate points of the lens whose two surfaces are represented by the two spheres shown.



The paraxial rays from O will refract on crossing the first surface as shown. Tracing these refracted rays backwards will show that they meet at $P^{/}$ which is at the image distance, s_{i1} , from the first surface's vertex V_1. We can use 7,17 on this first surface

NB Where the two circles overlap the refractive index is n_L the refractive index of the lens whilst elsewhere the refractive index is denoted n_m the refractive index of the medium the lens is placed in (usually air).

$$\frac{n_m}{s_{o1}}+\frac{n_L}{s_{i1}}=\frac{n_L-n_m}{R_1} \tag{7.23}$$

This first surface image acts as a virtual object for the second surface at a distance s_{o2} from that surface's vertex, V_2. It should be noted that as far as our preceding discussion is concerned the object space for the second refractive surface has a refractive index n_L . From examination of the diagram

$$|s_{o2}| = |s_{i1}| + d \qquad (7.24)$$

Where importantly we refer to the magnitudes of the distances (they have signs associated with them). Since s_{o2} is an object on the left hand side of vertex 2 it has a positive value and $s_{o2} = |s_{o2}|$ whereas the image relating to s_{i1} is also on the left and is therefore negative, $s_{i1} = -|s_{i1}|$

$$s_{o2} = -s_{i1} + d \qquad (7.25)$$

Therefore at the second surface the equation 7.17 yields

$$\frac{n_L}{-s_{i1} + d} + \frac{n_m}{s_{i2}} = \frac{n_m - n_L}{R_2} \qquad (7.26)$$

In this case $n_L < n_m$ and R_2 is negative making the RHS positive. This implies that the second term on the LHS must be larger than the first. Addition of 7.23 for the first surface and 7.26 for the second surface gives

$$\frac{n_m}{s_{o1}} + \frac{n_m}{s_{i2}} = (n_l - n_m)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{n_l d}{(s_{i1} - d)s_{i1}} \qquad (7.27)$$

Now the thin lens approximation is made where $d \to 0$

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{(n_l - n_m)}{n_m}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \left(\frac{n_l}{n_m} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad (7.28)$$

Where we have allowed the object of surface 1 to be ***the object*** and the image of surface 2 to be ***the image***.

Equation 7.28 is a central equation in the theory of lens known as the **Lens Makers Equation** or the **Thin Lens Equation**.

Were the lens to be in air (the most frequent case) then $n_m = 1$ and the lens makers equation becomes

$$\frac{1}{s_o}+\frac{1}{s_i}=(n_L-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \tag{7.28a}$$

7.28 can be used to find some special cases;

(i) If the object is at infinity, $s_o = \infty$, then the image will be formed at the focal plane, $s_i = f_i$ or image focus (by definition).

$$\frac{1}{f_i}=(n_L-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \tag{7.29}$$

(ii) Similarly moving the image to infinity will require the object to be at the object focus

$$\frac{1}{f_o}=(n_l-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \tag{7.30}$$

Or noting the symmetry between f_i and f_o

$$f_i = f_o = f \tag{7.31}$$

$$\frac{1}{f}=(n_L-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \tag{7.32}$$

(iii) Combining 7.28 and 7.32 we obtain

$$\frac{1}{f}=\frac{1}{s_o}+\frac{1}{s_i} \tag{7.33}$$

Equation 7.33 is known as the **Gaussian lens formula**. With equations 7.17, 7.28, 7.32 and 7.33 we can predict the behaviour of single lenses in most circumstances where the thin lens approximation holds good.

It is of interest and importance to note equation 7.31. This is telling us that the lens has the same focal length whichever way around it is facing!

Ie. an object at infinity will be focused at the same distance from the lens whichever side of the lens is towards the object.

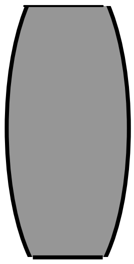
In applying these equations it is of greatest importance to apply the sign convention carefully!

For light propagating left to right	
Lens descriptors, R	**+ive if C is right of V**
Object descriptors s_o and f_o	**+ive left of V**
Image descriptors s_i and f_i	**+ive right of V**

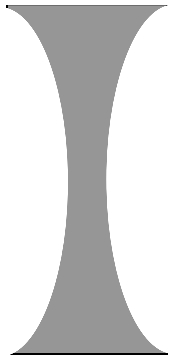
Looking at the previous diagram we can see how the convention applies to the situation depicted. The example chosen to study in detail was a bi-convex lens and we see that the centre of surface 1, C_1 is to the right of the vertex V_1 and therefore R_1 is positive. For surface 2, C_2 is to the lext of vertex V_2 and R_2 is therefore negative. s_{o1} is to the left of V_1 and is therefore positive whilst s_{i1} is also to the left of V_1 and is therefore negative. s_{o2} is to the left of V_2 and is therefore negative whilst s_{i1} is to the right of V_2 and is therefore positive.

We may look at a few examples:

CONVEX **CONCAVE**

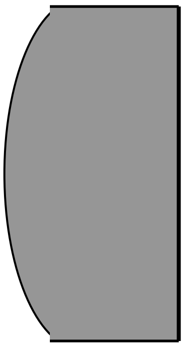


$R_1 > 0$
$R_2 < 0$

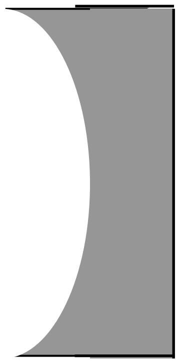


$R_1 < 0$
$R_2 > 0$

Bi-convex **Bi-concave**

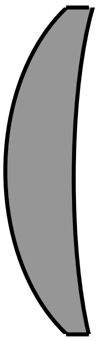


$R_1 > 0$
$R_2 = -\infty$

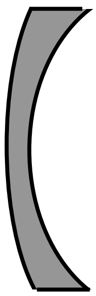


$R_1 < 0$
$R_2 = -\infty$

Plano-convex **Plano-concave**



$R_2 > R_1 > 0$



$R_1 > R_2 > 0$

Meniscus-convex **Meniscus concave**

Looking first at the bi-convex construction, the simplest structure may be with a symmetry such that $|R_1| = |R_2| = R$. The lensmakers equation is then for a lens in air

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f} = (n_L - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = (n_L - 1)\frac{2}{R} \qquad (7.28b)$$

Computation of the focal length of a thin lens is a straight forward matter using the lens makers equation with the important matter being to be taken care that the the sign convention is applied correctly. eg.

(i) For a bi-concave glass lens, n_l = 1.5, with radii of curvature $|R_1| = 10\text{cm}$ and $|R_2| = 15\text{cm}$ the focal length is found as follows;

For the first refractive surface R_1 is to the left of V_1 and is therefore negative, $R_1 = -10\text{cm}$, whilst for the second refractive surface R_2 is to the right of V_2 and is therefore positive, $R_2 = +15\text{cm}$

$$\frac{1}{f} = (1.5-1)\left(\frac{-1}{10} - \frac{+1}{15}\right)\text{cm}^{-1} = 0.5 \times \left(\frac{-3}{30} - \frac{2}{30}\right)\text{cm}^{-1} = 0.5 \times \frac{-5}{30} = \frac{-5}{60}\text{cm}^{-1}$$

$$f = \frac{-60}{5}\text{cm} = -12\text{cm}$$

(ii) For a plano convex glass lens with radius of curvature $|R_1| = 5\text{cm}$

The centre of first refractive surface R_1 is to the right of V_1 and is therefore positive whislt the centre of the second refractive surface is to the right (or left) of V_2 and we can call this positive or negative!

$$\frac{1}{f} = (1.5-1)\left(\frac{+1}{5} - \frac{\pm 1}{\infty}\right)\text{cm}^{-1} = 0.5 \times \left(\frac{+1}{5} \mp 0\right)\text{cm}^{-1} = 0.5 \times \frac{+1}{5} = \frac{+1}{10}\text{cm}^{-1}$$

$$f = +10\text{cm}$$

We can also use 7.28a to find the position of an image given the position of an object and the radii (or equivalently focal length) of a lens.

(iii) For a bi-concave glass lens with radii of curvature, R_1 , R_2 = -5cm and +10cm respectively. If the object is placed at 10 cm to the left of the lens, s_o = +10cm then the image distance, s_i , can be found using 7.28a

$$\frac{1}{+10}+\frac{1}{s_i}=(1.5-1)\left(\frac{-1}{5}-\frac{-1}{10}\right)$$

$$\frac{1}{s_i}\mathrm{cm}^{-1}=0.5\left(\frac{-2+1}{10}\right)-\frac{1}{10}=\frac{-1+0.5-1}{10}=\frac{-1.5}{10}$$

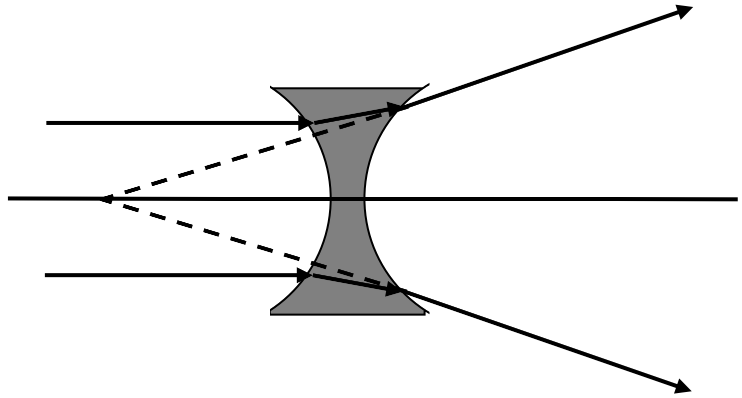
$$s_i=-\frac{20}{3}\mathrm{cm}=-6.666\mathrm{cm}$$

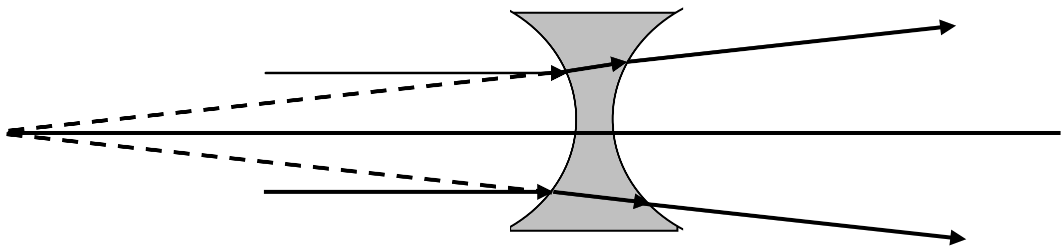
The negative sign tells us that the image is virtual and appears on the same side of the lens as the object, that is according to our convention, to the left of the vertices (recall we have a thin lens and the two vertices are for practical purposes coincident although strictly this refers to the vertex of the second surface).

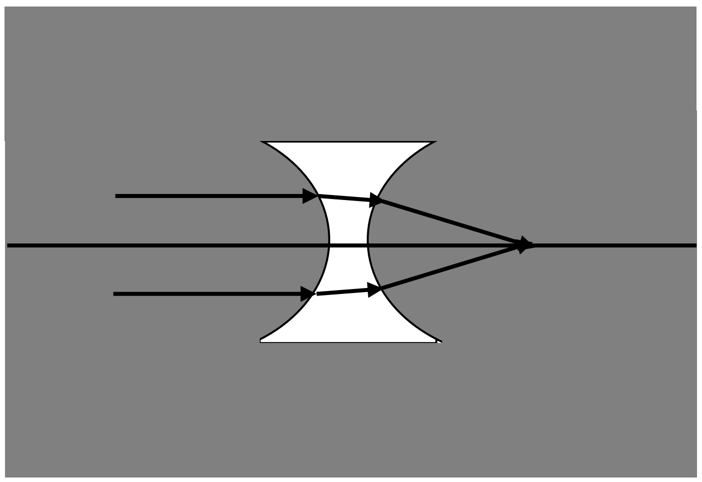
(iv) A final example is to find the focal length where the glass lens is as in **(i)** but is employed underwater. Water has a refractive index, $n_{H_2O}=1.33$

$$\frac{1}{f}=(\frac{1.5-1.33}{1.33})\left(\frac{-1}{10}-\frac{+1}{15}\right)\mathrm{cm}^{-1}=\frac{0.17}{1.33}\times\left(\frac{-3}{30}-\frac{2}{30}\right)\mathrm{cm}^{-1}=\frac{0.17}{1.33}\times\frac{-5}{30}=\frac{-0.85}{40}\mathrm{cm}^{-1}$$

$$f=\frac{-40}{0.85}\mathrm{cm}=-47.0\mathrm{cm}$$







(v) By making the difference between the refractive index of the lens and the medium smaller we have reduced the action of the lens and the object at infinity is imaged at a greater distance from the lens. It is still a virtual image. If the refractive index of the medium was greater than that of the glass lens, eg n_m = 2.5

$$\frac{1}{f} = (1.5 - 2.5)\left(\frac{-1}{10} - \frac{+1}{15}\right)\text{cm}^{-1} = -1.0 \times \left(\frac{-3}{30} - \frac{2}{30}\right)\text{cm}^{-1} = -1.0 \times \frac{-5}{30} = \frac{+5}{30}\text{cm}^{-1}$$

$$f = \frac{+30}{5}\text{cm} = +6\text{cm}$$

The bi-concave lens is now a positive, converging lens and will produce real images. Note in passing, the value of 2.5 for the refractive index of the medium is extremely high!

The situations (i), (iv) and (v) for the bi-concave lens as the medium refractive index is gradually increased until it is greater than that of the lens is depicted above in figures a to c.

The Gaussian lens formula, 7.33, may be re-written to make the image distance the subject for example

$$\frac{1}{f} = \frac{1}{s_o} + \frac{1}{s_i} \qquad (7.33)$$

$$\frac{1}{s_i} = \frac{1}{f} - \frac{1}{s_o} \qquad (7.33a)$$

$$s_i = \frac{s_o f}{s_o - f} \qquad (7.34)$$

Inspection of 7.34 shows that when the object is at infinity the image is at a distance +f.

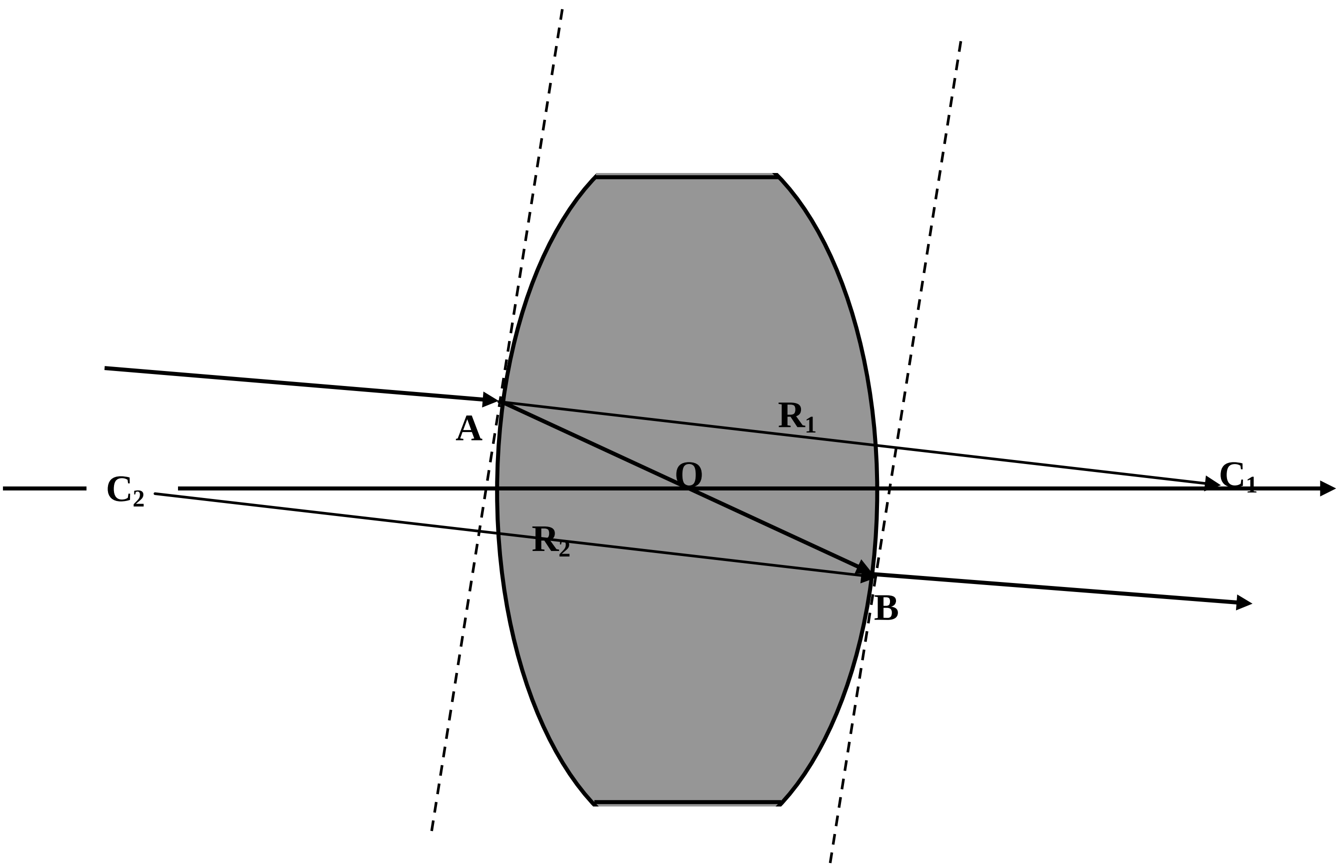
(i) ***<u>For a convex lens</u>*** where f is positive the image distance is positive if $s_o > f$ indicating a real image to the right of the lens. As s_o is reduced from infinity the denominator becomes smaller and s_i consequently larger. When the object is placed at $s_o = f$ the image is at infinity. Reducing the object distance yet further, so that $s_o < f$, will result in a negative value for s_i or a virtual image to the left of the lens.

(ii) ***For a concave lens*** if we inspect equation 7.34, for all positive object distances, $s_o > 0$, ie. all real objects to the left of the lens, the denominator is positive whilst the numerator is negative. Thus the image distance is negative ie. it is virtual and to the right of the lens. At $s_o = \infty$ the image is at $-|f|$. As the object distance is reduced from infinity the numerator approaches zero from a negative value whilst the denominator approaches a limiting value of +f (for s_o = 0) s_i therefore approaches zero from a negative value (-f when $s_o = \infty$). To obtain a real image with this lens we need a virtual object!

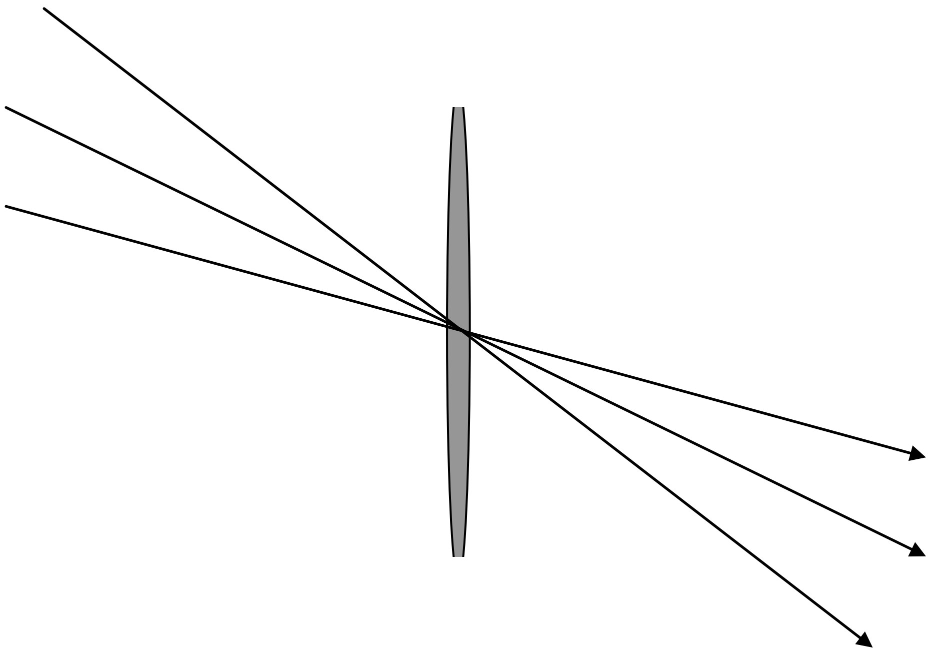
Focal planes, focal points and ray tracing.

Ray tracing is an extremely effective way of finding out how an optical system will behave. We have noted in what has preceded that the axial ray because it impinges upon and exits the surfaces normally is not displaced in any way, in agreement with refraction at a plane interface. Are there any other special rays that we can use?

The construction shown below shows the straight through axial ray. It also picks points A and B, one on each surface and constructs the tangent through each where A and B are chosen such that the two tangents are parallel to one another. The radii, R_1 and R_2 through points A and B and centres, C_1 and C_2 of the two surfaces are also shown.



The lines BC_2 and AC_1 are parallel and therefore the two triangles, AC_1O and BC_2O are similar. This means that $\frac{R_1}{R_2} = \frac{OC_1}{OC_2}$. As the values of R and the positions of the centres are fixed this means that the position of O is a fixed point independent of where A and B are chosen and it is defined as the ***optical centre*** of the lens. The ray AOB can be seen as equivalent to a ray passing through two plane parallel surfaces and there will be no angular displacement, only a slight lateral displacement (exaggerated in the diagram) which tends to zero in the thin lens approximation.



We can therefore draw a ray from any point in object space through the centre of the lens and it will be an undeviated straight line. We then have two rays that we can draw immediately in any ray construction. We also know that any ray parallel to the axis must be deflected through the focal point and conversely that any ray through a focal point must emerge parallel to the axis.

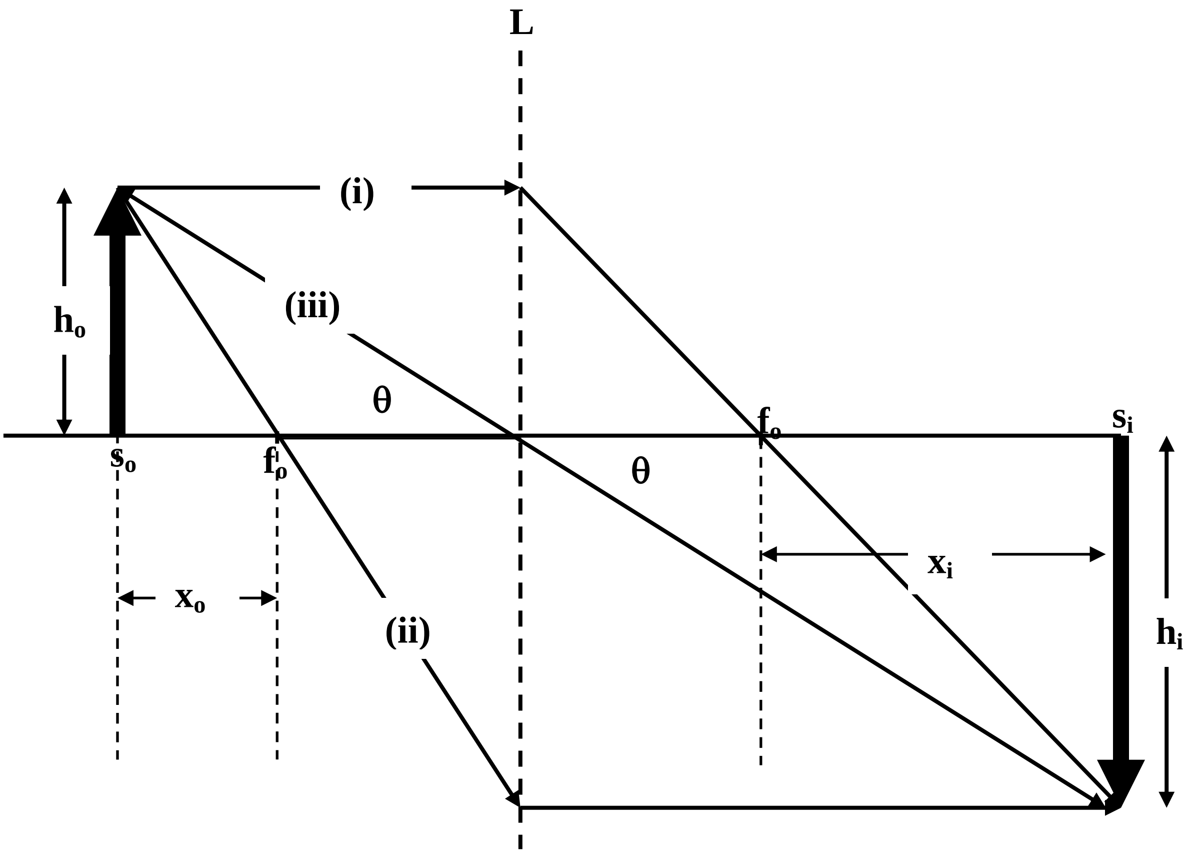
The three rays we have available to use in a construction beside the optical axis are ;

- **(i)** A ray from the object parallel to the optical axis will be refracted at the lens through the image focus.
- **(ii)** A ray from the object through the focus will be refracted parallel to the optical axis.
- **(iii)** A ray from the object through the optical centre of the lens will pass undeviated.

NB. Rays (i) and (ii) may involve the construction of a virtual ray by back tracing.
Two examples of ray tracing are given in the following two diagrams;

(i) The converging lens.

Representing a thin converging lens by the dashed line perpendicular to the axis and representing the object by a thick arrow, an example of ray tracing is shown below.The diagram shows the positive (converging)lens at L with the rays (i), (ii) and (iii) included



The above construction includes all of the principle rays and from it we learn a number of things. The image to object height is given by

$$\frac{|h_i|}{|h_o|} = \frac{|s_i|}{|s_o|} = |M_T| \tag{7.35}$$

And M_T is known as the transverse magnification. There is a sign convention to be observed when discussing transverse magnification. The convention requires that transverse distances measured above the optical axis are taken as positive whilst those below are negative. In this example both s_i and s_o are real and positive but h_o is positive whilst h_i is negative and therefore M_T is negative.

Using the previously established relation between object and image distance, 7.33 , $s_i = \dfrac{s_o f}{s_o - f}$.

$$M_T = \frac{f}{s_o - f} \tag{7.36}$$

<u>Newtonian equation for thin lens.</u>

An alternative way of defining distances was used by Newton. Taking the distances of object and image as measured from the focal point such that

$$x_o = s_o - f \qquad \text{and} \qquad x_i = s_i - f \tag{7.37}$$

Examining the above diagram, on the LHS of the lens we see two similar right hand triangles, one above and one below the axis with f in common

$$\frac{h_o}{h_i} = \frac{x_o}{f} \tag{7.38a}$$

Similarly on the RHS

$$\frac{h_i}{h_o} = \frac{x_i}{f} \tag{7.38b}$$

Combining these

$$\frac{f}{x_o} = \frac{x_i}{f} \tag{7.39}$$

and

$$x_o x_i = f^2 \tag{7.40}$$

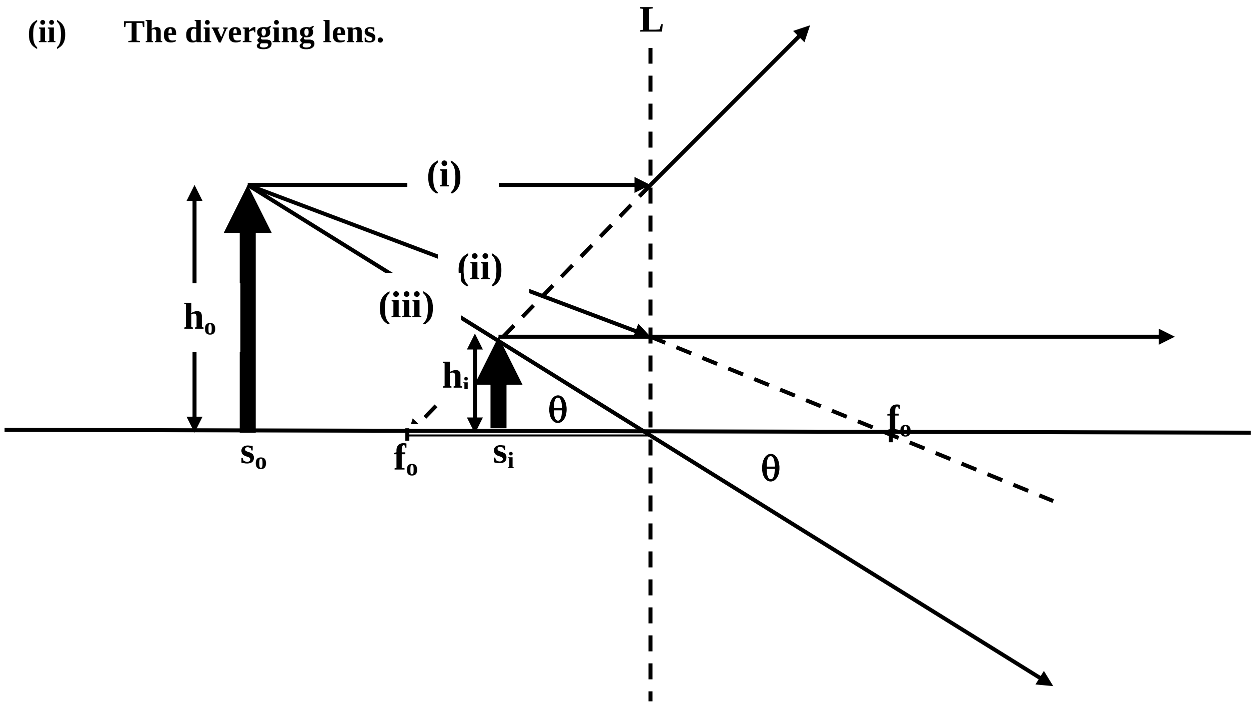
Equation 7.40 is the Newtonian form of the Gaussian lens equation.

An axial or longitudinal magnification may be defined using 7.40 as the rate of change of imageaxial distance with object axial distance

$$M_L = \frac{dx_i}{dx_o} = -\frac{f^2}{x_o^2} = -M_T^2 \tag{7.41}$$

Equation 7.41 holds some interesting implications regarding the action of a lens in that if the transverse magnification is doubled by for example reducing the object distance or x_o the longitudinal magnification is increased by ×4. There is in other words an apparent change in perspective. A good example of this is the foreshortening effect seen in a telephoto lense where when focusing on two or more distant objects, the further object would not be judged as being as far away as it is if the lateral size/height of the object were the only clue, eg. two men in the distance , one twice as far away as the other, would appear to differ in height by only ×1.44 (their actual height being equal) giving the false perspective that the further man was much closer to the nearer man than he actually is.

(ii) The diverging lens.



The above construction uses the same three rays (as with a converging lens at L) but now with a diverging lens at L. The dashed lines are projections of the rays and the image which is formed to the left of the lens is a virtual image.

Again, the transverse magnification can be defined

$$\frac{h_i}{h_o} = \frac{s_i}{s_o} = M_T$$

Both object and image are upright and therfore h_o and h_i have the same sign and M_T is positive.

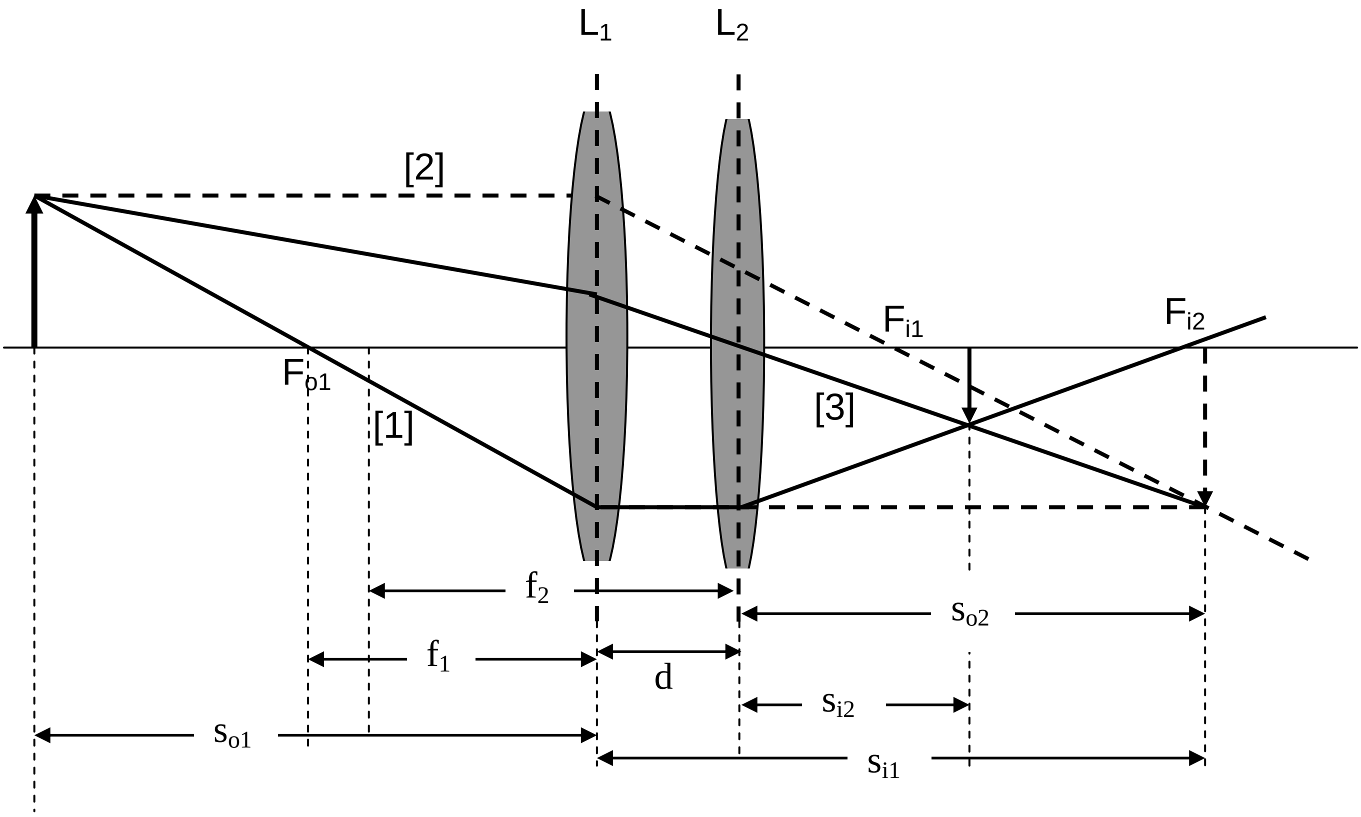
$$M_T = \frac{f}{s_o - f} = \frac{|f|}{s_o + |f|}$$

Compound thin lenses.

Many optical instruments involve the creation of a compound lens from two or more lenses. Many camera lenses involve four and more lenses acting together. In such a setup the analysis can be complex and is often carried out using a computer although it may be useful to use ray tracing etc,. as an initial means of finding the type of compound structure capable of achieving a particular purpose. To keep things simple we will look at the use of two thin lenses used together as a single compound lens. In carrying out this analysis we begin by splitting the problem of a pair of lenses used together into two possible situations.

i) <u>When the separation between the two lenses is less than either focal length.</u>

We begin by considering two positive thin lenses, L_1 and L_2 separated by a distance d smaller than either focal length. We may find the resultant image by ray tracing using similar techniques as previously employed for the single lens as follows.



Begin by ignoring L_2 and construct the diagram as usual for L_1 with

i) A ray **[1]** passing through F_{o1} and then refracting at L_1 parallel to the optical axis.

ii) A ray **[2]** travelling parallel to the optic axis in the object space and being refracted through F_{i1}

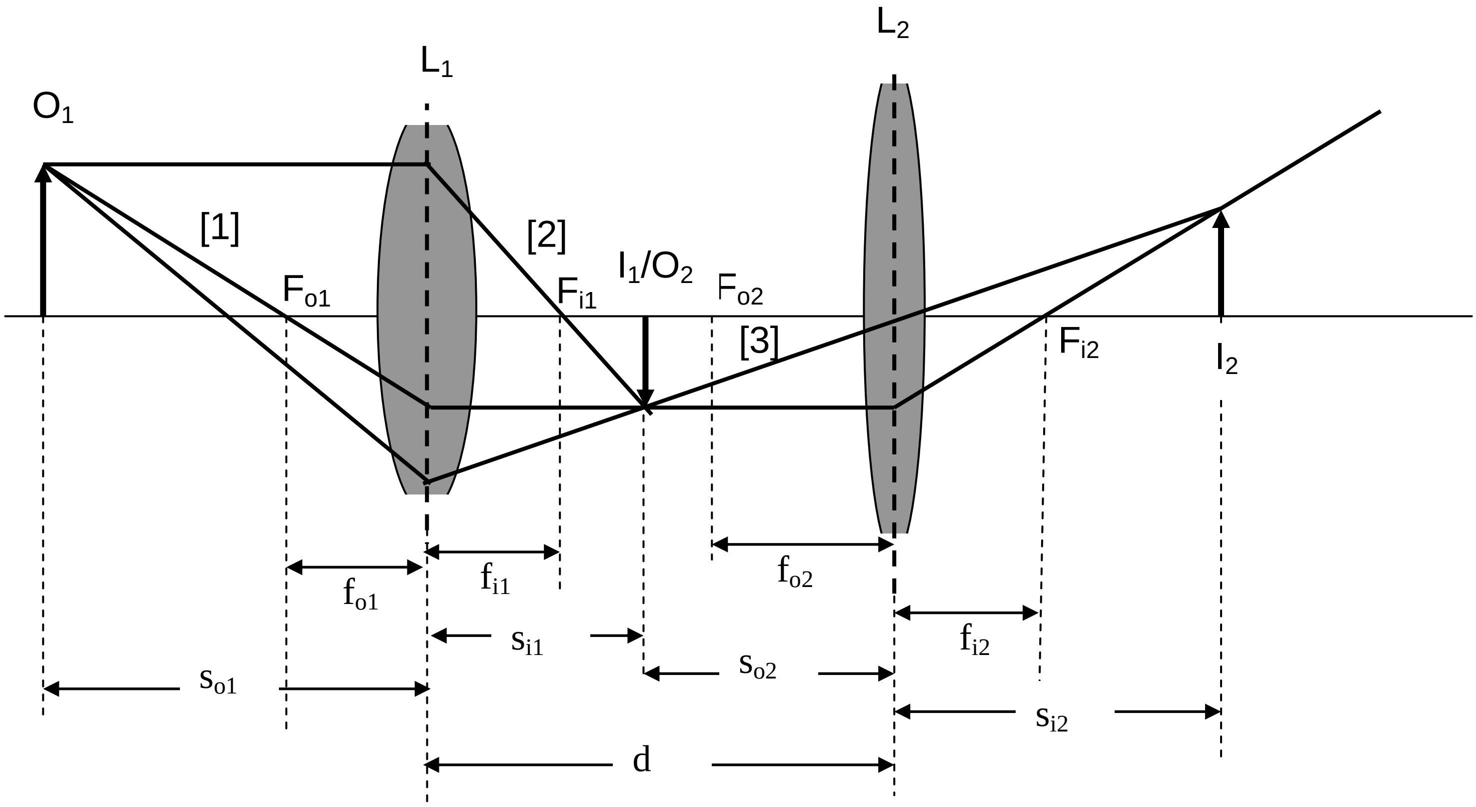
The previous two rays locate the top of the first attempt image that now acts as the object for L_2.

iii) Now draw a ray **[3]** back from the top of the image/object through the optical centre of L_2 and allow it to be refracted by L_1 to intercept the top of the object.

The above diagram shows the actual rays **[1]** and **[3]** in bold and the other rays used in the construction as dashed lines (when only L_1 mattered). These rays define the actual image position. It is inverted and demagnified in this case.

ii) <u>When the separation between the two lenses is greater than either focal length.</u>

As for the previous analysis we begin by finding the image for L_1 alone and then use this as the object for L_2 to find the image due to both lenses



i) Construct a ray **[1]** passing through F_{o1} and then refracting at L_1 parallel to the optical axis.

ii) Construct a ray **[2]** parallel to the optic axis passing through F_{i1} the intersection of **[1]** and **[2]** give the image of L_1 which now acts as the object O_2 of L_2

iii) Construct a ray **[3]** from the top of the image I_1/O_2 through the centre of L_2 acting as the undeviated ray through the centre of L_2 and extend this backwards to L_1 where it is refracted back to the top of the object O_1

We have a final image for the two lens system, I_2. It is not inverted

We can analyse this using the Gaussian lens formula sequentially on L_1 and L_2 beginning with L_1 and finding the position of its image s_{i1}.
We begin by noting that as far as the compound lens system is concerned the actual object distance is $s_o = s_{o1}$ and the actual image distance is $s_i = s_{i2}$
Applying GLF to L_1

$$\frac{1}{s_{i1}} = \frac{1}{f_1} - \frac{1}{s_{o1}} = \frac{1}{f_1} - \frac{1}{s_o}$$

$$s_{i1} = \frac{s_o f_1}{s_o - f_1}$$

And for L_2 the image from L_1 acts as the object. We need to find this object distance as follows

$$\mathrm{s_{o2} = d - s_{i1}}$$

If $\mathrm{d > s_{i1}}$ then the object for L_2 is real whereas it is virtual when $\mathrm{d < s_{i1}}$

$$\frac{1}{s_{i2}} = \frac{1}{s_i} = \frac{1}{f_2} - \frac{1}{s_{o2}}$$

rewritten as

$$s_i = \frac{s_{o2} f_2}{s_{o2} - f_2}$$

We have already found s_{o2} and using this obtain

$$s_i = \frac{(d - s_{i1}) f_2}{d - s_{i1} - f_2}$$

We can obtain a more useful expression if everything is ***in terms of the first object distance s_o*** and the ***second image distance s_i*** which are **<u>the object</u>** and **<u>the image</u>** distances of the compound system by using the equation relating s_{i1} and s_{o1} through f_1 ,

$$s_{i1} = \frac{s_o f_1}{s_o - f_1}$$

$$s_i = \frac{f_2 d - f_2 \, {}^{s_o f_1}\!\!\diagup_{(s_o - f_1)}}{d - f_2 - {}^{s_o f_1}\!\!\diagup_{(s_o - f_1)}}$$

The distance of the last optical surface of a system to the second focal point of that system is known as the ***back focal length b.f.l*** and the distance of the first optical surface from the first focal point or object focus is known as the ***front focal length, f.f.l***

i) ***front focal length, f.f.l***

From Gaussian lens formula **if we allow** $s_i \to \infty$ then $s_{o2} \to f_2$ or $s_{i1} \to d - f_2$

In this case the Gaussian lens formula gives

$$\left.\frac{1}{s_o}\right|_{s_i \to \infty} = \frac{1}{f_1} - \frac{1}{d - f_2} = \frac{d - (f_1 + f_2)}{f_1 (d - f_2)}$$

Of course this value of s_o is the *f.f.l* ie the position in front of the compound lens system at which an object is placed to cause an image to be formed at infinity just as would be the case for a single lens

$$f.f.l = \frac{f_1(d - f_2)}{d - (f_1 + f_2)}$$

ii) ***back focal length b.f.l***

In the same way **if we allow** $s_o \to \infty$ then $s_o - f_2 \to s_o$ and in this case

$$\left.\frac{1}{s_i}\right|_{s_o \to \infty} = \frac{1}{f_2} - \frac{1}{d - f_1} = \frac{d - (f_1 + f_2)}{f_1(d - f_2)}$$

$$s_i = b.f.l = \frac{f_2(d - f_1)}{d - (f_1 + f_2)}$$

The *b.f.l* is the distance from the compound lens system at which the image is formed when the object is at infinity just as would be the case for a single lense.

We note that unlike a single lens where $f_o = f_i$ in the case of the compound lens system the $b.f.l \neq f.f.l$ and it is of importance which way around the lens system is facing except in a special case.

If the lenses are brought into contact and $\mathrm{d} \to 0$ then we find

$$\mathrm{b.f.l} = \mathrm{f.f.l} = \frac{\mathrm{f_2 f_1}}{\mathrm{f_2 + f_1}}$$

Otherwise stated

$$\frac{1}{\mathrm{f}} = \frac{1}{\mathrm{f_1}} + \frac{1}{\mathrm{f_2}}$$

Important equations and topics in analysis of lenses

1) ***Paraxial approximation***

2) ***Lens makers equation***

$$\frac{1}{s_o}+\frac{1}{s_i}=(n_l-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

3) ***Sign convention***

4) ***Gaussian lens formula***

$$\frac{1}{f}=\frac{1}{s_o}+\frac{1}{s_i}$$

5) ***Combining lens maker plus Gauss***

$$\frac{1}{f}=(n_l-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \qquad \Rightarrow \qquad \frac{1}{f}=\left(\frac{n_l}{n_m}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

6) ***Ray tracing***

7) ***Transverse magnification***

$$\frac{|h_i|}{|h_o|}=\frac{|s_i|}{|s_o|}=|M_T|$$

8) ***Longitudinal magnification***

$$M_L=\frac{dx_i}{dx_o}=-\frac{f^2}{x_o^2}=-M_T^2$$

9) ***Newtonian form of Gaussian equation***

$$x_o x_i=f^2$$

10) ***Compound lens formula for lenses in contact***

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}$$

11) ***Front focal length and back focal length for lens pair***

$$f.f.l = \frac{f_1(d - f_2)}{d - (f_1 + f_2)} \qquad \mathrm{b.f.l} = \frac{\mathrm{f}_2(\mathrm{d} - \mathrm{f}_1)}{\mathrm{d} - (\mathrm{f}_1 + \mathrm{f}_2)}$$

12) ***Compound lens formula for two lenses separated by a distance d***

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

Optical Instruments.

i) Vision Defects and Corrective Optics

At this point, before discussing vision defects and their correction with lenses it is useful to introduce the concept of the ***dioptric power*** of a lens. We have seen a number of equations where the quantity $\frac{1}{f}$ was the subject of the equation eg for the relation between a curved surface and the focal length of that surface we found

$$\frac{1}{f} = (n_l - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

And for two lenses or refracting surfaces in contact the combined focal length was

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$$

The quantity $\frac{1}{f} = \mathcal{D}$ where $\mathcal{D}$ is the dioptric power of the lens, a quantity used by optometricians.

We then see that the dioptric power of a lens is

$$\mathcal{D} = (n_l - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

If we consider a bi-convex (or bi-concave) to be two plano convex (plano concave) lenses placed in intimate contact then we may speak of the dioptric power of one of the surfaces

$$\mathcal{D}_1 = (n_l - 1)\frac{1}{R_1} \qquad \text{and also} \qquad \mathcal{D}_2 = -(n_l - 1)\frac{1}{R_2}$$

where we have used $R_1 = \infty$

And the dioptric power of the lens is simply

$$\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$$

The sign convention is as always to be taken into account.

For a double lens combination when in contact

$$\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are the dioptric powers of the two lenses seperately.

The usefulness of this quantity is apparent and is of particular convenience for optometrical purposes where the eye lens plus spectacle lens system are used in conjunction to correct vision defects.

The eye with its lens is able to alter the focal point of the lens by muscle control over the curvature of the lens. The relaxed eye with a large radius of curvature will have a greater focal length or smaller dioptric power and it is usually as a relaxed system that objects at infinity are focussed onto the retina. Bearing in mind that the image distance is fixed at the retina nearer objects will require that the focal length is reduced according to the Gaussian lens formula

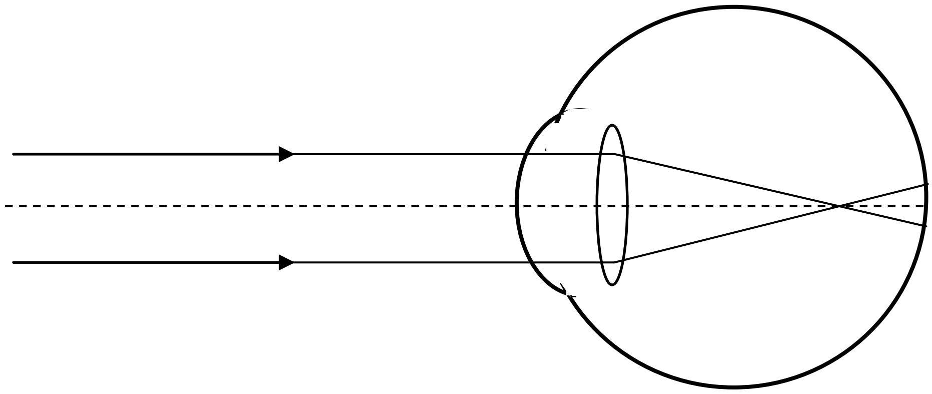
$$\frac{1}{\mathrm{f}} = \frac{1}{\mathrm{s_o}} + \frac{1}{\mathrm{s_i}} \qquad \Rightarrow \qquad \frac{1}{f} = \frac{1}{s_o} + C$$

This is possible to a certain extent but there will be a point called the near point and it is not possible to focus any object closer to the eye than this. The near point becomes further from the eye as we age and is why we tend, for example, to hold books further away in order to read comfortably as we age. Eye correction is possible with an appropriate combination of spectacle lens and eye lens.

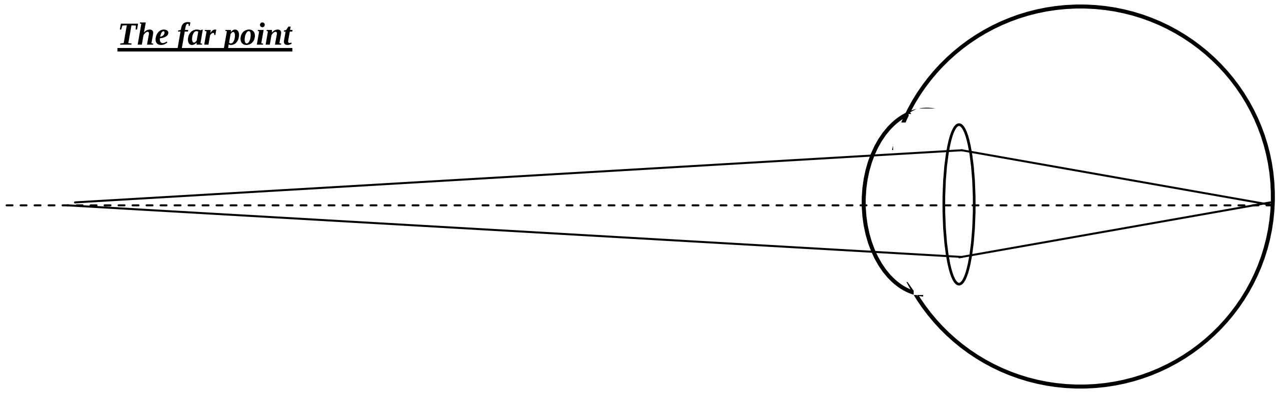
a) Shortsightedness (Myopia)

One of the most common vision defects is shortsightedness or myopia. This is where the focal length of the relaxed eye, unaided is shorter than the distance between lens and retina ie. an object at infinity is focused somwhere in front of the eyeball. As the object approaches the eye the image will be formed closer to the retina until the far point is reached where the image is formed at the retina by the relaxed eye. Any closer and the eye may accommodate to allow the image to continue to be formed on the retina. In short the diotric power of the relaxed lens is too great (it converges too much).

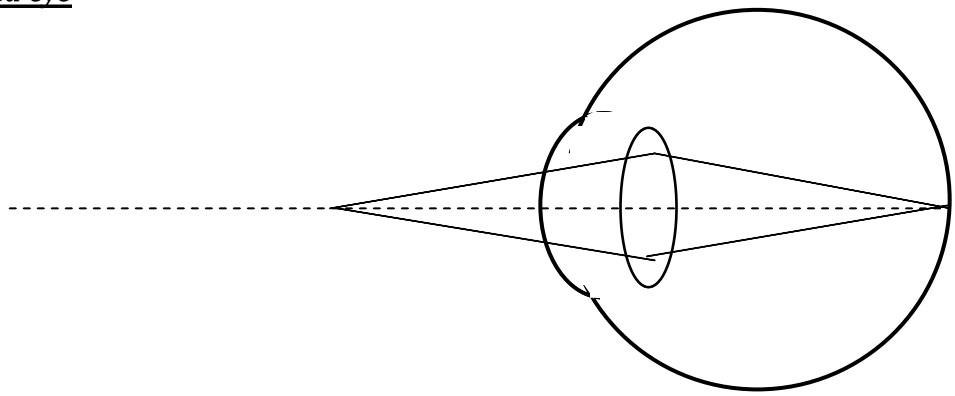
Object at infinity



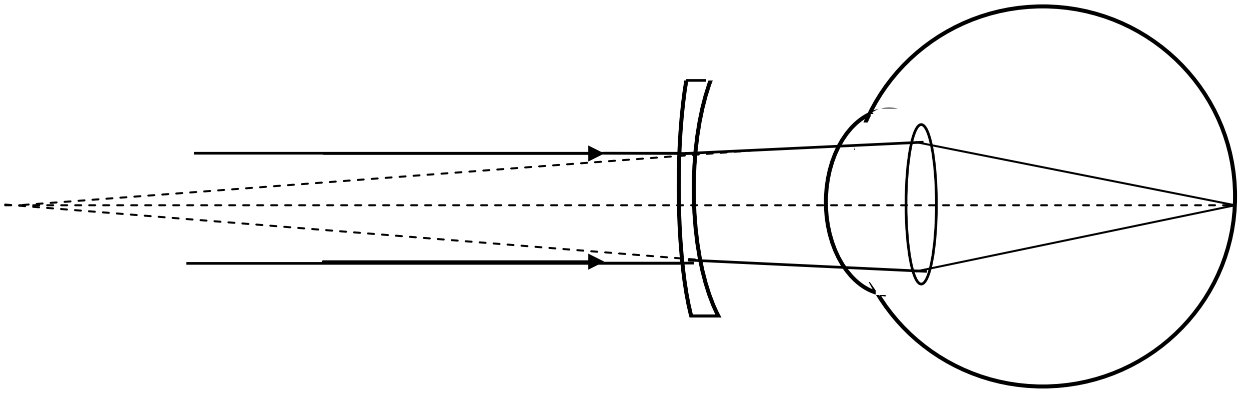
The far point



The accomodated eye



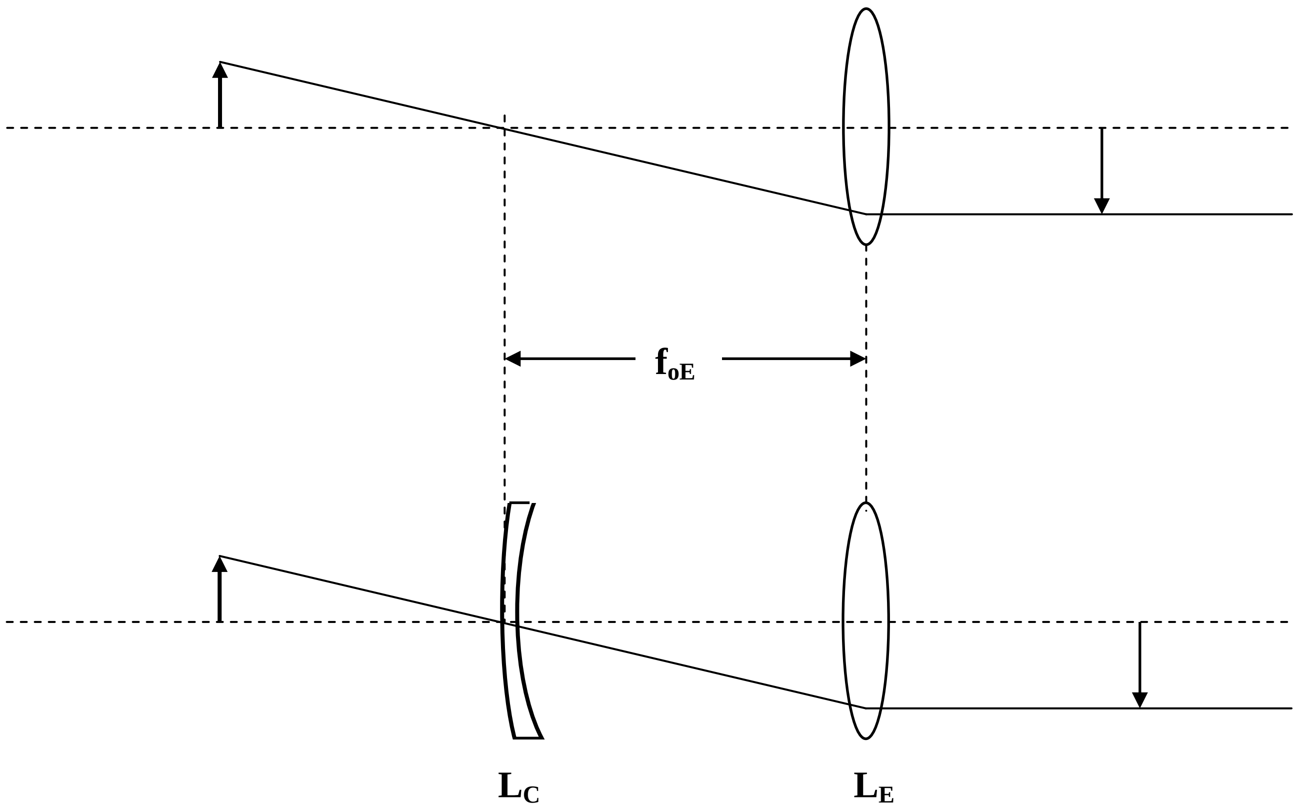
Given that the dioptric power of the relaxed myopic eye is too great (and positive) and that $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$ we can correct this problem by introducing a lens with a negative dioptric strength (diverging lens) with the effect of reducing $\mathcal{D}$



A meniscus lens with a surface whose concave radius is smaller than its convex radius will serve to correct myopia as the overall dioptric strength of these two surfaces will be negative. The negative lens is chosen such that it forms a virtual image of an object at infinity at the far point which is able to be focussed by the relaxed eye. eg. If the myopic has a far point at 2 metres then the lens is chosen such that

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{\infty} - \frac{1}{2m} = \frac{1}{f_L} \qquad f_L = -2m \qquad \mathcal{D} = -0.5m^{-1}$$

We need to note now that this is ok for contact lenses where the lens and contact lens are separated by a very small, effectively zero, distance. We have seen that the actual distance between two lenses forming a compound system frequently has to be accounted for. The separation is usually chosen such that the spectacles are placed at the front focal point of the relaxed eye lens. This is done in order to ensure that the unaided eye has the same magnification as the corrected eye. Often the far point of each eye would be different and care must be taken to ensure that there is no difference in magnification between the two eyes as this would create greater problems for the user. The diagram below illustrates that the placement of the corrective lens at f_{oE} and how this does not alter the magnification of the unaided eye



Knowing the focal length f_L of the corrective lens and the focal length f_E of the eye we may find the back focal length which should be the distance from eye lens to retina

$$b.f.l = \frac{f_E(d - f_L)}{d - (f_L + f_E)}$$

We may compare the power of a spectacle lens with that of a contact lens correcting the same condition by using for the contact lens/eye lens system

$$\frac{1}{f} = \frac{1}{f_C} + \frac{1}{f_E}$$

This combined focal length and the b.f.l need to be equal to the eye lens retina distance and

$$\frac{f_C f_E}{f_C + f_E} = \frac{f_E(d - f_L)}{d - (f_L + f_E)}$$

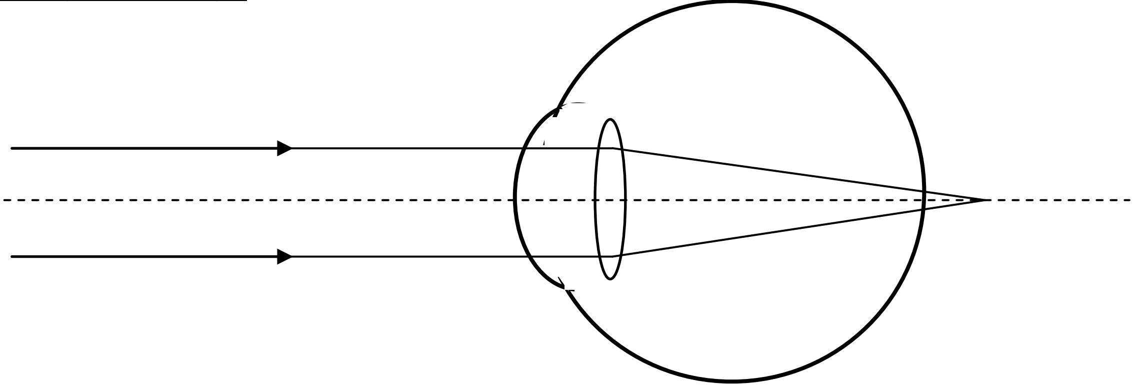
Simplifying and re phrasing

$$\frac{1}{f_C} = \frac{1}{f_L - d}$$

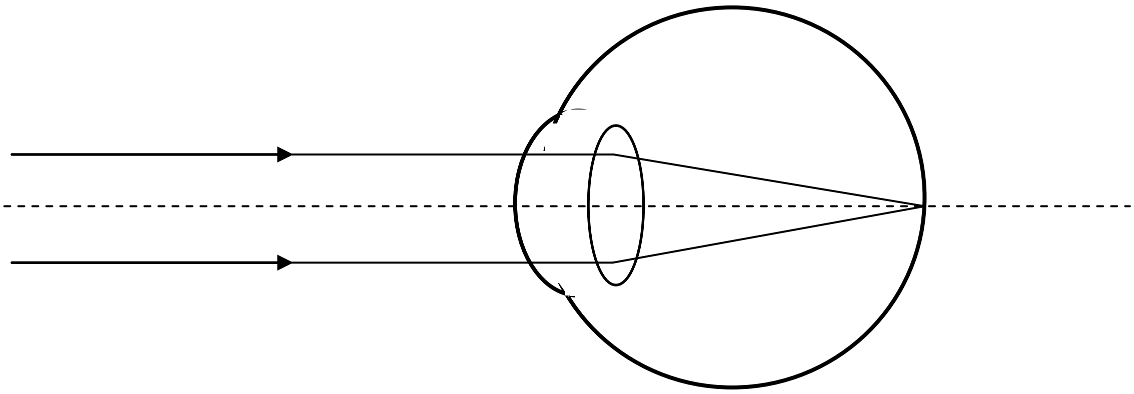
b) Longsightedness (Hyperopia)

Where the focal length of the relaxed eye, unaided is longer than the distance between lens and retina ie. an object at infinity is focused somwhere beyonf the retina the sufferer requires reading glasses to correct hyperopia. This is the complementary condition to myopia and we can begin by constructing the diagrams that we had previously in our consideration of that defect

<u>Object at infinity, relaxed eye</u>

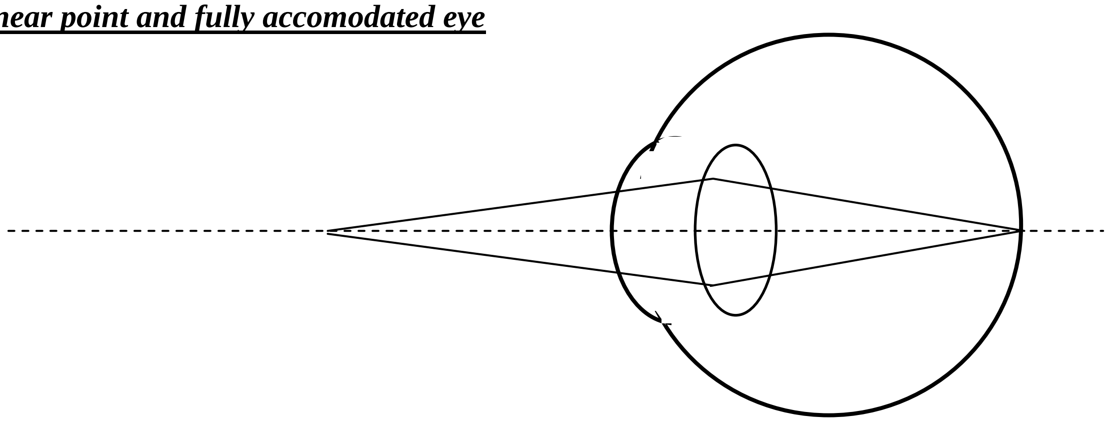


Object at infinity, accomodated eye



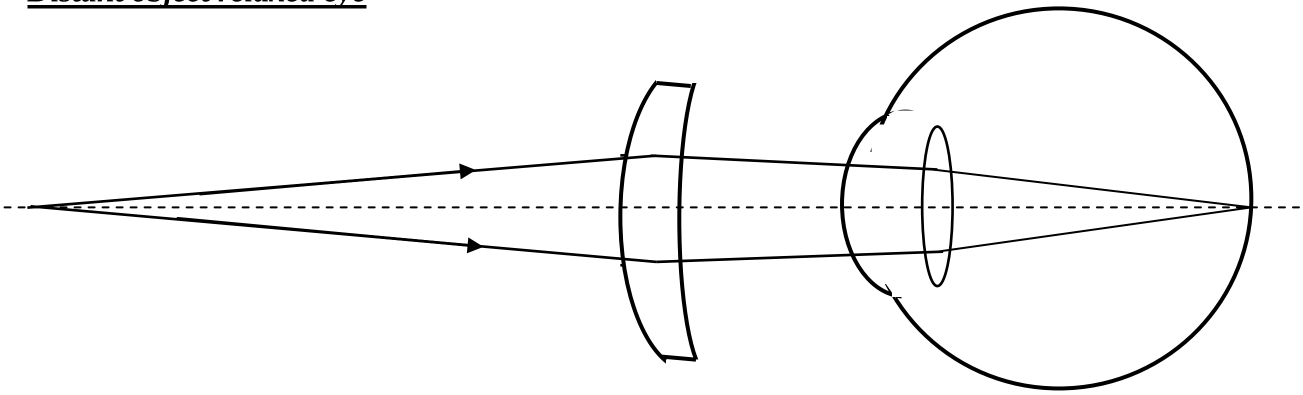
The eye may form an image of a distant object by accomodating but this will only be possible until the object is at the near point

The near point and fully accomodated eye

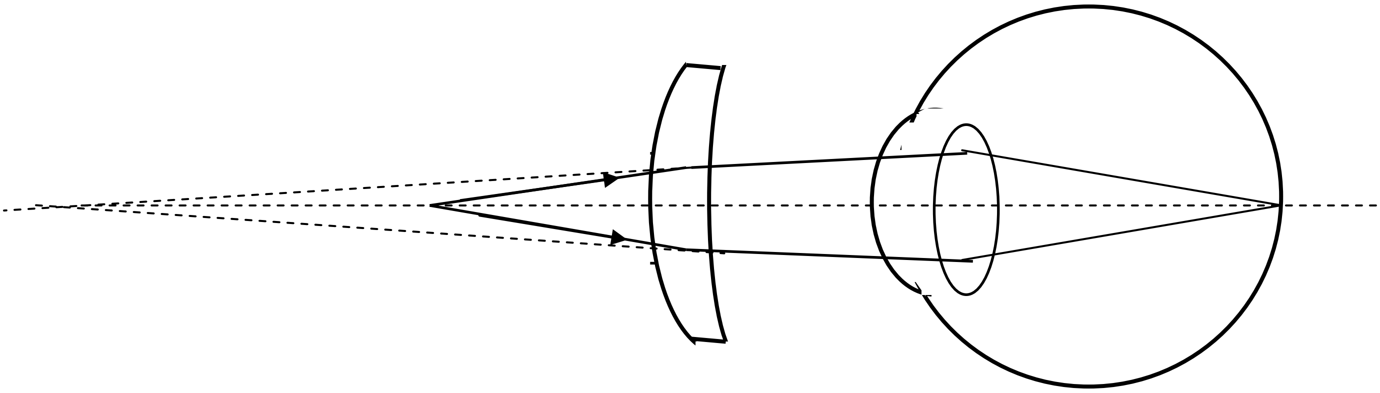


The lens focuses (accomodates) to form an image at the retina of an object at the near point. Any closer and the unaided hyperopic eye will be unable to accommodate in order to form an image. To correct this defect and enable the eye to focus objects closer than the near point requires that the lens converges and that the eye/lens pair have a greater dioptic power than the unaided eye. This requires that the lens is convex or more usually a meniscus lens is used with a surface whose concave radius is greater than its convex radius. Such a lens will serve to correct for an object closer than the near point of the unaided eye by producing a virtual image further from the eye than the near point that acts as a virtual object beyond the near point.

Distant object relaxed eye



Nearby object accomodated eye



The above diagrams illustrate the action of a lens correcting hyperopia. An example will further aid understanding. Suppose the hyperopic eye has a near point of 100cm and an object eg a book is held at 50cm. It is then required that the corrective lens form a virtual image of this object (the print of the book) at 100cm to act as a virtual object for the eye.

$$s_o = 50cm \qquad\qquad s_i = -100cm$$

Using the Gaussian lens formula we can find the required focal length of the corrective lens

$$\frac{1}{f_L} = \frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{50} - \frac{1}{100} = \frac{1}{100}$$

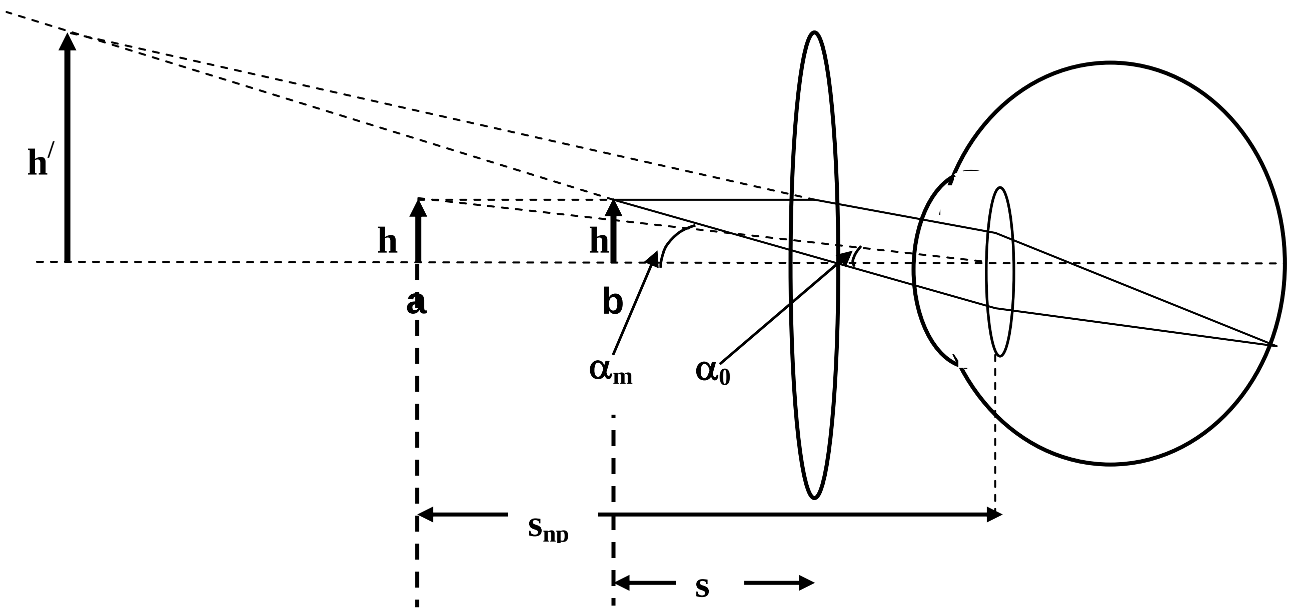
$$f_L = 100cm$$

To form the virtual image object must be closer to the positive lens than the focal distance as is the case here.

ii) Magnifying Glass

When viewing an object the image at the retina may be increased in size thus increasing the perceived size by moving the object closer. However as the eye cannot focus on an object closer than the near point (usually about 25cm from the eye) at position a in the diagram where there is maximal accomodation of the eye lens there is a natural limit to how large the perceived image may become with the unaided eye alone. To obtain a larger image a further lens may be used.

The magnifying glass allows magnification of an object that is positioned closer to the lens than the focal length of that lens and closer than the near point of the eye at position b in the figure. It achieves this by creating an erect virtual image to act as an object for the eye.



The above diagram illustrates the working principle of the magnifying glass where the virtual image of height $h^{/}$ is viewed by the eye as an erect virtual object. Normally the viewer will adjust the position of the lens, eye and object for comfortable viewing. Clearly the linear transverse magnification produced will depend on these relative positions and ***not on the particulars of the lens used.*** The angular magnification may be defined as the ratio of the angles α_m and α_0 the angle subtended at the eye by the magnified virtual object and the angle subtended at the eye by the object at the near point.

$$M_\alpha = \frac{\alpha_m}{\alpha_0} = \frac{h/s}{h/s_{np}} = \frac{s_{np}}{s} \approx \frac{25cm}{s(cm)}$$

At one extreme if the virtual image/object is viewed at infinity then $s = f$ and

$$M_\alpha = \frac{25cm}{f(cm)}$$

The other extreme is where the image is viewed at the near point and the image distance

$$s_i = -25cm$$

From the Gaussian lens formula

$$\frac{1}{s} - \frac{1}{25} = \frac{1}{f} \qquad \Rightarrow \qquad s = \frac{25f}{25+f}$$

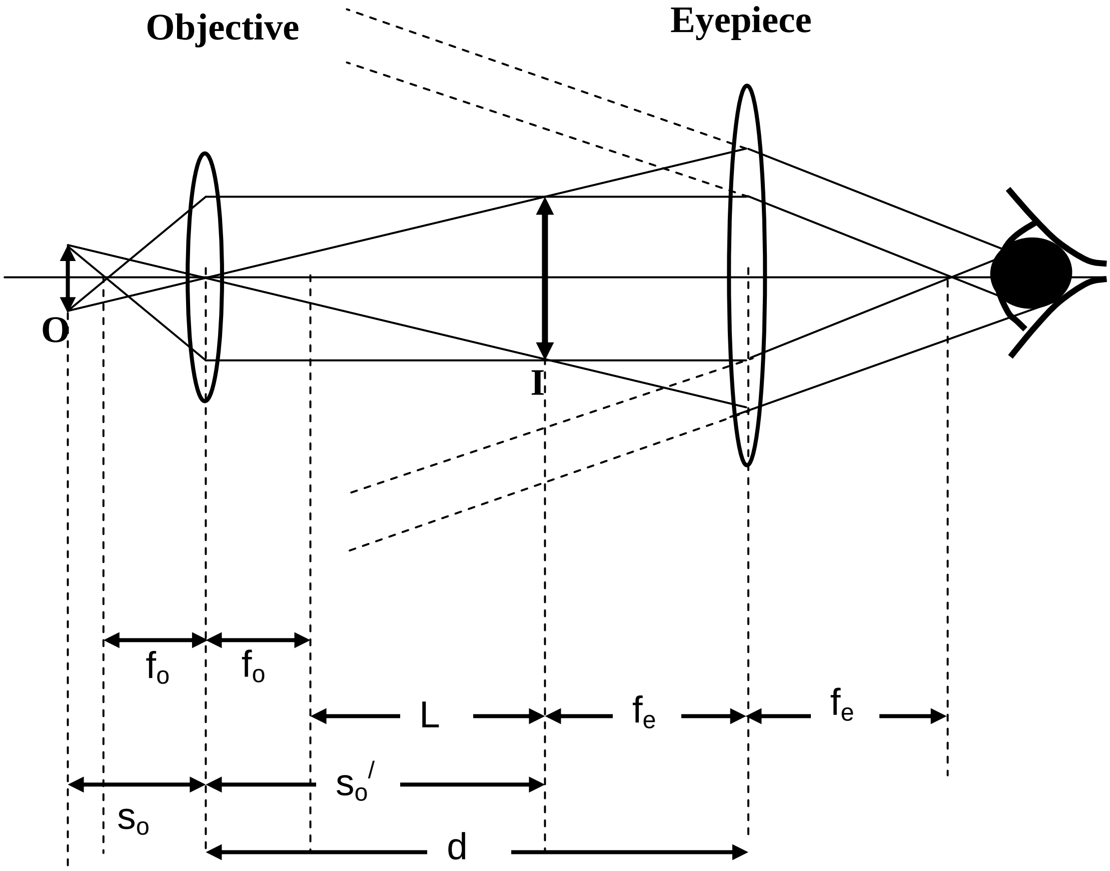
Giving an angular magnification at the near point of

$$M_\alpha = \frac{25}{25f}(25+f) = \frac{25cm}{f(cm)} + 1$$

The actual magnification will lie between these two values as the viewer adjusts position.

iii) Microscope

It is usually the case that magnifiers are used to aid the eye in viewing images formed by optical components of another optical system. The optical microscope is a good example of this type of use.



The compound microscope increases the possibility of magnification obtained with the single positive lens (magnifying glass) where now an eyepiece is used to view and magnify the image created by an initial lens or objective of very short focal length. The basic setup is shown above.

The angular magnification is given by the same equation as for the magnifying glass with the exception that the effective focal length of the objective/eyepiece combination replaces the focal length of the single lens.

$$M_\alpha = \frac{25cm}{f_{eff}}$$

The effective focal length of two lenses separated by distance d is given approximately by

$$\frac{1}{f_{eff}} = \frac{1}{f_o} + \frac{1}{f_e} - \frac{d}{f_o f_e}$$

$$\frac{1}{f_{eff}} = \frac{f_e}{f_o f_e} + \frac{f_o}{f_o f_e} - \frac{d}{f_o f_e} = \frac{f_e + f_o - d}{f_o f_e}$$

cf the front focal length $f.f.l = \dfrac{f_1(d - f_2)}{d - (f_1 + f_2)}$

$$\frac{1}{f.f.l} = \frac{1}{f_{eff}} = \frac{d - (f_1 + f_2)}{f_1(d - f_2)} = \frac{d - (f_o + f_e)}{f_o(d - f_e)} = \frac{f_e + f_o - d}{f_o f_e - df_o}$$

Substituting we have for the magnification

$$M_\alpha = \frac{25cm}{f_{eff}} = \frac{25(f_o + f_e - d)}{f_o f_e}$$

We can use the Gaussian lens formula and the fact that $s_o^{/} = d - f_e$, to find that

$$\frac{s_o^{/}}{s_o} = \frac{d - f_o - f_e}{f_o}$$

and with both of these equations together

$$M = -\left(\frac{s_o^{/}}{s_0}\right)\left(\frac{25}{f_e}\right)$$

Ie the total magnification is the product of the linear transverse magnification of the objective multiplied by the angular magnification of the eyepiece when viewing the final image at infinity.

We can also use the form for the transverse magnification as given using the Newtonian formulation in terms of x_o and x_i to obtain

$$|m| = \left|\frac{h_i}{h_o}\right| = \left|\frac{s_o^{/}}{s_o}\right| = \left|\frac{x^{/}}{f_o}\right| = \left|\frac{L}{f_o}\right|$$

And thus find a more useful form for the total magnification

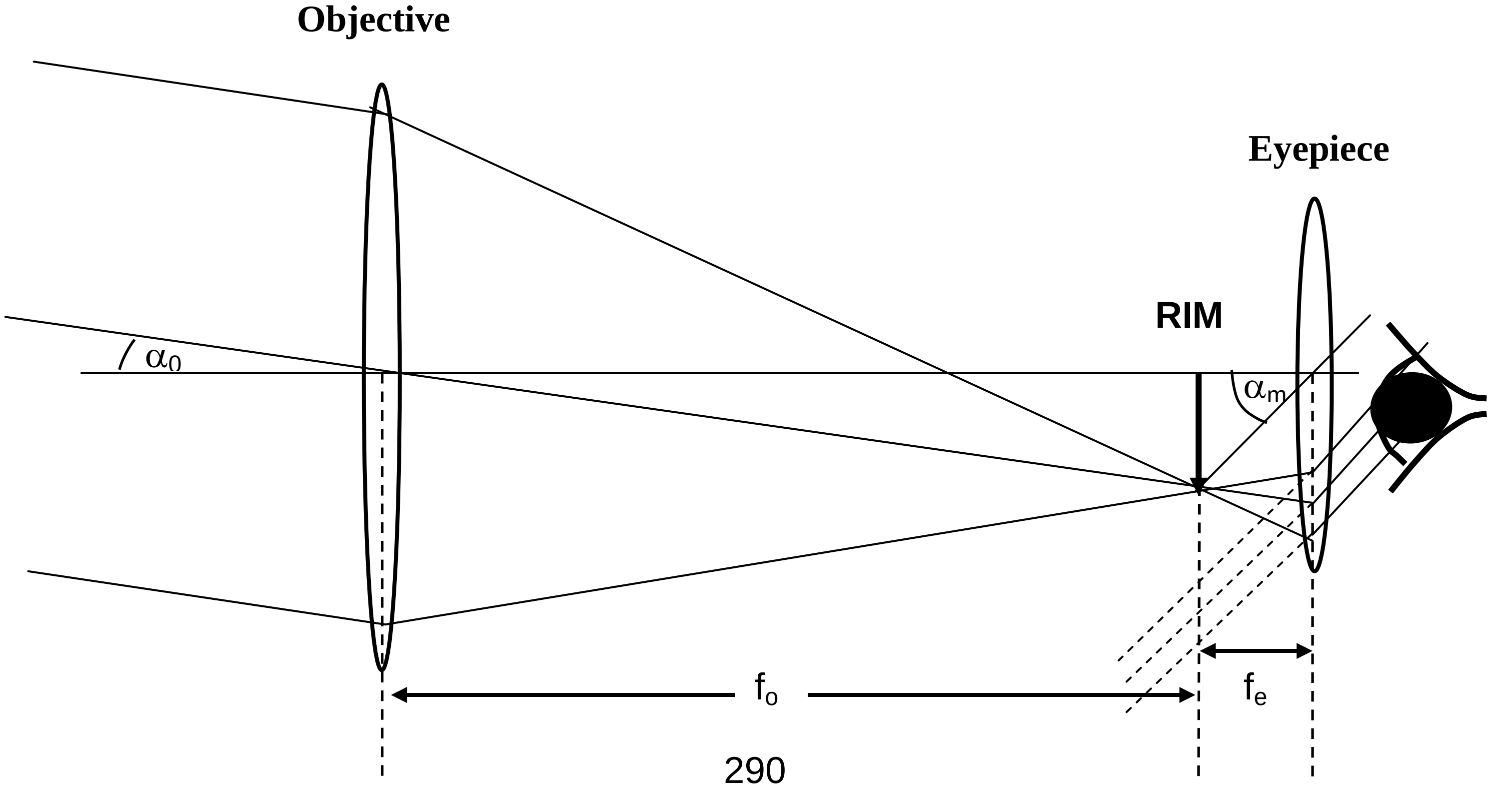
$$M_\alpha = -\left(\frac{25}{f_e}\right)\left(\frac{L}{f_o}\right)$$

The total length of the microscope is $f_o + f_e + L$

Finally we should note that the eyepiece and objectives may be multi-element lens systems themselves. Any professional microscope will be a great deal more complex than the one described by the diagram here although the principle of operation is the same.

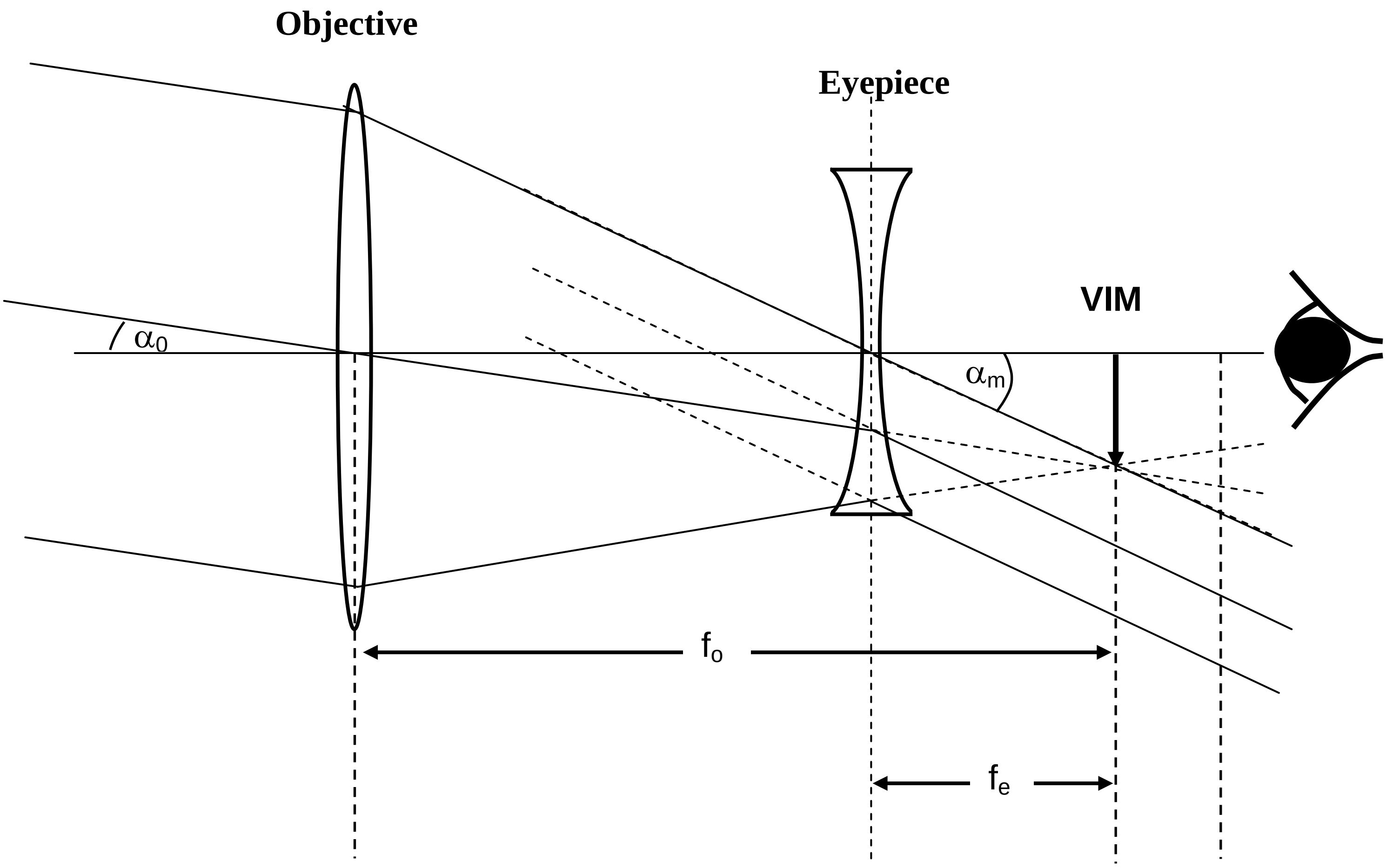
iv) Telescope

The refracting telescope is designed to obtain angular magnification of objects at large distances effectively at infinity.



The principles of the astronomical or **Keplerian telescope** are shown with reference to the above diagram. The important feature that marks this configuration of two lenses out is that the focal points of the objective and the eyepiece coincide between the two positive lense. This requires that the separation between the two lenses is given by $d = |f_o| + |f_e| = f_o + f_e$. ***The objective forms a real intermediate imageRIM*** at the focus of the eyepiece for which it acts as a real object. The real intermediate object being at the focus of the eyepiece will result in parallel beam emerging from the system. ***The final image is in this case inverted.*** This is identical to the beam expander that we have already considered but with the long focal length lens in front.

The alternative refracting telescope configuration is known as the **Galilean telescope** as shown in the diagram below.



This configuration consists of a positive (converging) objective lens followed by a negative (diverging) eyepiece lens. Again the focal points of the two lenses coincide but this time the focal length of the eyepiece is negative and the coincident foci appear after the eyepiece. Here the lens separation is $d = |f_o| + |f_e| = f_o - f_e$. ***The virtual intermediate image, VIM*** in this case is a virtual object for the eyepiece. ***The final image is erect.***

From the right angle triangles formed by the intermediate image, the undeviated ray and optic axis in both configurations we can see that the angular magnification is given in both cases by

$$M_\alpha = \frac{\alpha_m}{\alpha_0} = -\frac{f_o}{f_e}$$

Bearing in mind the sign of f_e for each configuration we have a negative value for M_α for the first Keplerian telescope consistent with the inverted image and a positive value of M_α for the Galilean telescope again consistent with the non-inverted image in this case.