ELECTROMAGNETIC WAVES & OPTICS.

PREFACE.

The objective of the course is to provide an account of the study of optics taking up the subject from where the study of electric and magnetic fields finished with the elucidation of the Maxwell equations and developing these into a wave description of light. Some basics of the properties of electromagnetic waves will be studied allowing an exploration of physical optics. Geometric optics will then provide an understanding of simple optical components and the use of such components in some common optical instruments. The course will end by exploring some of the quantum aspects of light and how this affects the description of the interaction of light with matter.

1. MAXWELLS EQUATIONS & the WAVE EQUATION

a) Maxwell's equations in free space.

The starting point of this course is where the first year electromagnetism course ended, namely with the four Maxwell's equations and the electromagnetic wave equation. We therefore begin with a review of these equations.

In free space (no charges or currents) there are a set of four equations, established by Maxwell, that relate the magnetic field and the electric field. These equations are written in differential form as:

$$\nabla \bullet E = 0 \qquad (1.1a)$$

$$\nabla \bullet B = 0 \qquad (1.2a)$$

$$\nabla \times E = -\frac{dB}{dt} \qquad (1.3a)$$

$$\nabla \times B = +\mu_0 \varepsilon_0 \frac{dE}{dt} \qquad (1.4a)$$

and are symmetric in form. ε_0 is the electric permittivity of free space (= 8.85×10^{-12} Fm^{-1}) and μ_0 is the magnetic permeability of free space (= $4\pi \times 10^{-7}$ NA^{-2})

We often need to use the equations in the form that includes the field sources where ρ is the charge density in free space and J is the current density, $= \frac{i}{A}$ where i is the current. ρ and J are source terms for the electric and magnetic fields. The equations then become;

$$\nabla \bullet E = \frac{\rho}{\varepsilon_0} \qquad (1.1b)$$

$$\nabla \bullet B = 0 \qquad (1.2b)$$

$$\nabla \times E = -\frac{dB}{dt} \qquad (1.3b)$$

$$\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{dE}{dt} \qquad (1.4b)$$

Maxwells equations are often written in integral form using Gauss's Law or Stoke's Law for the divergence and curl equations respectively:

$$\oint \vec{E}.d\vec{A} = \frac{Q}{\varepsilon_0} \qquad (1.1c)$$

$$\oint \vec{B}.d\vec{A} = 0 \qquad (1.2c)$$

$$\oint \vec{E}.d\vec{l} = -\frac{d\phi_B}{dt} \qquad (1.3c)$$

$$\oint \vec{B}.d\vec{l} = \mu_0 \left(i_C + \frac{d\phi_E}{dt} \right) \qquad (1.4c)$$

where ϕ_E and ϕ_B are the electric and magnetic flux through the area A enclosed by the loop (= EA and BA respectively) and Q is the charge enclosed by the surface implied in 1.1c.

Electromagnetic Wave Equation

We may use equations 1a – 4a to derive a wave equation in the absence of charge and current densities and in vacuum which describes all electromagnetic waves propagating in free space. We can eliminate one of the two dependent variables (E or B) by using the vector identity

$$\nabla \times (\nabla \times \Psi) = \nabla(\nabla \bullet \Psi) - \nabla^2 \Psi \qquad (1.5)$$

where Ψ is any vector field.

For instance, using 1.3a, 1.4a (and 1.1a for $\nabla \bullet E = 0$)

$$\nabla^2 \vec{E} = -\nabla \times (\nabla \times \vec{E}) = \nabla \times \left(\frac{\partial \vec{B}}{\partial t} \right) = \frac{\partial}{\partial t}(\nabla \times \vec{B}) = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E}}{\partial t^2} \right) \qquad (1.6)$$

This is recognisably the wave equation of the form

$$\nabla^2 \Psi = \frac{1}{v^2} \frac{\partial^2 \Psi}{\partial t^2} \qquad (1.7)$$

Thus the electromagnetic wave equation has been derived for fields obeying Maxwell's equations;

$$\nabla^2 E = \varepsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} \qquad (1.8)$$

where

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \qquad (1.9)$$

is the speed of light in vacuum and given our values for ε_0 and μ_0, c = 2.99 $\times 10^8$ ms^{-1}.

Solutions of the Wave Equation

The fact that we obtain this wave equation from Maxwell's field equations tells us that the electric field has wave like solutions with its magnitude varying with space and time in a related way. The same is true of the magnetic field as we could have derived a wave equation for the magnetic field using a similar approach to that taken for the electric field. In fact the two fields, more precisely their time and spatial variation, sustain one another each acting as a source for the other. The original sources are the charge and current densities.

We are familiar with some solutions to such an equation and perhaps the most common is the plane wave solution.

a) Plane Wave Solutions.

There are two common ways of writing a so-called plane wave solution to the equation. The first of these is the **<u>cosinusoidal notation</u>**;

$$\vec{E}(z,t) = \vec{A}_0 \sin\left[2\pi\left(\frac{z}{\lambda} + \nu t\right)\right] = \vec{A}_0 \sin[kz + \omega t] \qquad (1.10)$$

The important parameters describing the plane wave are;

λ = wavelength which is the spatial period within which the wave repeats.

ν = linear frequency (often written f) which is the reciprocal of the temporal period within which the wave repeats, ie. $\nu = \dfrac{1}{T}$ where T is the period of the wave.

To compact the plane wave equation we can also specify;

k = wavevector , $\dfrac{2\pi}{\lambda}$, (in this defining equation the vector character is suppressed as we have stipulated that the wave is travelling in the z direction).

ω = angular frequency, $2\pi\nu$,

$\vec{A}_0$ = amplitude.

We easily see that in 1.10 if we increase z by $m\lambda$, where m is an integer, the field is unchanged, similarly of we increase the time by mT the field is unchanged. In 1.10 we have made an arbitrary choice such that the field is zero when z and t are both zero. This may not be possible when predicting the effect of superposition of two fields if their peaks and troughs do not coincide and in this case a further factor may be added to 1.10 called the phase shift, δ, such that

$$\vec{E}(z,t) = \vec{A}_0 \sin\left[2\pi\left(\frac{z}{\lambda} + \nu t\right) + \delta\right] = \vec{A}_0 \sin[kz + \omega t + \delta] \qquad (1.10a)$$

Using 1.10 as a trial solution in the wave equation 1.8 we obtain

$$-k^2 \vec{A}_0 \sin[kz + \omega t] = \frac{-1}{c^2} \omega^2 \vec{A}_0 \sin[kz + \omega t] \qquad (1.11)$$

This gives us a relation between c, k and ω;

$$c = \frac{\omega}{k} \qquad (1.12)$$

Using a cosine in place of the sine is the trivial matter of a phase difference of $\frac{\pi}{2}$.

Note the vector character of the amplitude as it represents the electric field, itself a vector. This is important when we come to discuss polarisation.
For a plane wave travelling in a general direction, <u>***r***</u> we may rewrite 1.10

$$\vec{E}(\vec{r},t) = \vec{A}_0 \sin\left[\vec{k} \bullet \vec{r} + \omega t\right] \qquad (1.13)$$

The <u>**exponential notation**</u> for waves is also very useful and for ease of manipulation is frequently preferred.

$$\vec{E}(\vec{r},t) = \vec{A}_0 \exp\, j\left[\vec{k} \bullet \vec{r} + \omega t\right] \qquad (1.14)$$

where $j = \sqrt{-1}$. The alternative notation $i = \sqrt{-1}$ is often used but we use i for currents occasionally later in the course and will stick with the j notation for imaginary numbers.
By choosing one of these notations and differentiating wrt position and wrt time we find

$$\frac{\partial E}{\partial r} = jk\vec{A}_0 \exp\, j\left[\vec{k} \bullet r + \omega t\right] \qquad (1.15)$$

and

$$\frac{\partial E}{\partial t} = j\omega \vec{A}_0 \exp\ j\left[\vec{k} \bullet \vec{r} + \omega t\right] \qquad (1.16)$$

Therefore dividing 1.16 by 1.15 we find the velocity

$$c = \frac{dr}{dt} = \frac{\omega}{k} = \nu\lambda = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \qquad (1.17)$$

as found in 1.12.

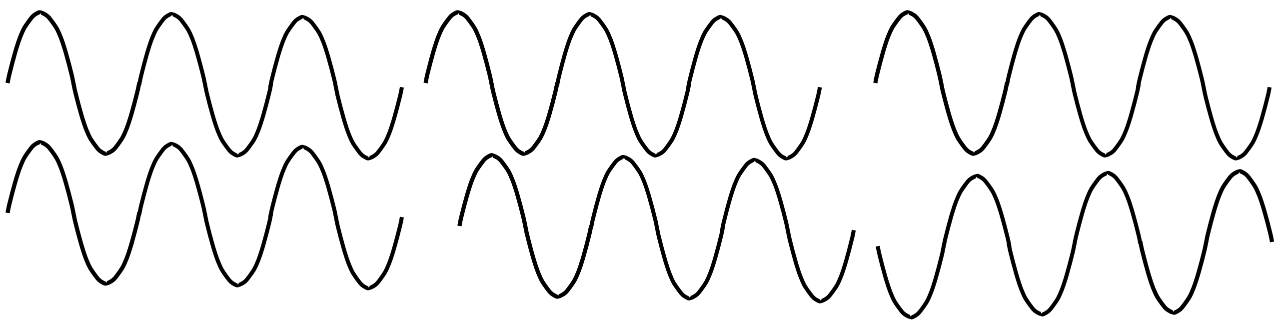
The phase, ϕ , of a wave is often a useful property and whether using the cosinusoidal or exponential notation the phase is the argument of the cosine or exponential, ie.

$$\phi = \vec{k} \bullet \vec{r} + \omega t + \delta \qquad (1.18)$$

For a plane wave travelling in the z direction

$$\phi = kz + \omega t + \delta \qquad (1.18b)$$

Its importance will become apparent later in the course but 1.18 or 1.18b should be recalled. The phase is the information in the plane wave equation telling us the relative position of one wave wrt another, ie. where the peaks and troughs of one wave are wrt another of the same wavelength. The phase shift δ may be different for each wave or of course zero.



In Phase $\qquad$ 90^0 out of phase $\qquad$ 180^0 out of phase.

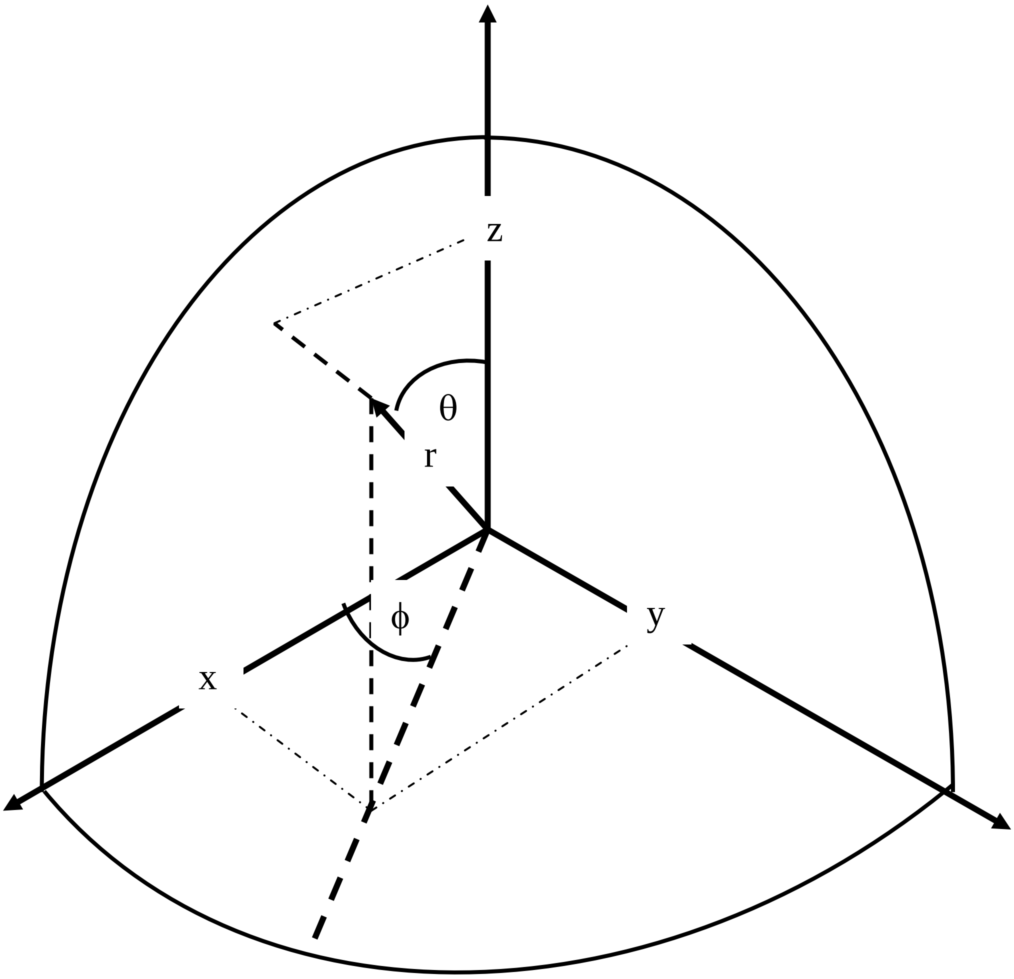
In the above diagram the phase shift of the three upper waves can be chosen as $\delta = 0$. For the waves beneath them, on the left the phase is also $\delta = 0$, in the centre $\delta = -90^0$ and on the right $\delta = \pm 180^0$.

A plane wave is so called because, for example , a plane wave travelling in the *z* direction has a constant wavefront represented by the *xy* plane. That is, the amplitude does not depend on *x* or *y* at a particular position *z* and point in time. The *xy* plane in that case forms a constant phase front and at any particular value of *z* across the *xy* plane the waves will all have the same relative phase.

Plane waves are therefore highly idealised and will not exist as a physical reality. However, for the monochromatic light beam from a highly collimated laser, the plane wave will be a superb approximation as such sources are highly directional and the power/intensity in the beam (which we shall see is proportional to the square of the field amplitude) is substantially independent of distance travelled.

b) Spherical Waves.

Point sources of light act as a source of spherical waves where the constant phase front is a sphere, centred on the point source, that expands in time at the speed of light, the radius of curvature of the sphere being $r = ct$ after a time t. To describe such waves mathematically we need to use spherical co-ordinates, r, θ, ϕ in place of cartesian co-ordinates, x, y, z.



$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \qquad (1.19)$$

In spherical co-ordinates

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2} \tag{1.20}$$

In our case of spherical waves the *E* and *B* fields have no dependence on θ or ϕ as they are spherically symmetrical and the differentials wrt these variables are then zero. This is of course why we choose to use spherical polar co-ordinates in this particular case because with this symmetry we obtain a simple Laplacian

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} \tag{1.20a}$$

This allows us to write;

$$\nabla^2 E = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rE) \tag{1.20b}$$

This may be checked by double differentiating the quantity *rE* wrt *r* and showing that 1.20b gives the same operation on *E* as would 1.20a

We do this because then the wave equation, 1.8, takes on a particularly useful form

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rE) = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} \tag{1.21}$$

Multiplying both sides by *r*

$$\frac{\partial^2}{\partial r^2}(rE) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(rE) \tag{1.22}$$

This is the simple one dimensional wave equation where the spatial variable is *r* and the field is replaced by the product of field and spatial variable and this means that the solution is simply

$$rE(r,t) = f(r-ct) \qquad \Rightarrow \qquad E(r,t) = \frac{f(r-ct)}{r} \tag{1.23}$$

This represents a wave propagating outwards from a point at a velocity *c*. There is a second solution representing a wave travelling inwards towards a point

$$E(r,t) = \frac{g(r+ct)}{r} \tag{1.24}$$

This expression diverges at *r* = 0 and is therefore unphysical.

The general solution would be a sum of the two

$$E(r,t) = C_1 \frac{f(r-ct)}{r} + C_2 \frac{g(r+ct)}{r} \tag{1.25}$$

A special type for this solution is the Harmonic Spherical Wave solution.

$$E(r,t) = \frac{A}{r} cos\big(k(r \mp ct)\big) \tag{1.26}$$

$$E(r,t) = \frac{A}{r} \cos\big(kr \mp \omega t\big) \tag{1.26a}$$

$$E(r,t) = \frac{A}{r} \exp\big(kr \mp \omega t\big) \tag{1.26b}$$

It is noteworthy that in the case of the spherical wave above, in contrast to that of the plane wave, 1.12, there is a dependence of the amplitude on distance travelled and as a result the intensity of the light (power per unit area proportional to the amplitude squared) falls as reciprocal square of the distance from the point source. This is a well known law and comes about as a result of the constant energy from the point source being shared out over an increasing area, $4\pi r^2$, as the wave travels outward.

We note that we began this by constraining the wave to be spherically symmetric. Other types of solution are possible but will not interest us further.

Maxwell's equations in simple media.

Maxwell's Equations

So far we have chosen to interest ourselves in the simple problem of electromagnetic waves in free space ie vacuum. Most of this course will be concerned with electromagnetic waves in media. We will now consider simple media such as glasses and liquids which are;

(i) **Homogeneous;** the same everywhere under translation i.e. the wave velocity, wavevectors and frequency which must be defined from place to place remain unchanged and

(ii) **Isotropic;** the same in all directions, i.e. under rotation. This means that the field vectors which have directionality by virtue of their vector nature are not affected by the medium except in the simplest way or put in another way the polarisation of the wave is unaffected by the medium.

The modification to what has already been done is straightforward. Free space is homogeneous and isotropic so ***the new thing here is that the electric and magnetic fields may interact with the medium and its charges.*** We may attribute simple scalars to account for the electric and magnetic properties of the medium. The simplest way to do this is to note that ***the action of the electric and magnetic fields associated with a light wave on a medium is to generate an electric or magnetic polarisation in that medium.*** That is, when an electric field, such as that associated with the electromagnetic wave, is present there may be a separation of positive and negative charge in the medium leading to a polarisation, ***P*** per unit volume and ***P***, in homogeneous and isotropic media is related to the electric field by

$$\vec{P} = \varepsilon_0 \chi_E \vec{E} \tag{1.27}$$

where χ_E is the **electronic susceptibility**

Polarisation per unit volume and electric field are also related to the electric displacement vector ***D*** sometimes used in place of the electric field which is defined as follows;

$$\vec{D} = \varepsilon\varepsilon_0 \vec{E} = \varepsilon_0 \vec{E} + \vec{P} \tag{1.28}$$

ε is the relative permittivity also known as the dielectric constant of a medium and thus

$$\varepsilon = 1 + \chi_E \qquad (1.29)$$

D is a useful way of representing the electric field in the presence of a medium.
In an analogous manner the magnetic field ***H*** is often used when dealing with magnetic media in the place of the magnetic induction, ***B***

Analagous to the electric field we have a magnetisation induced in magnetic materials where the magnetic field of the light wave interacts with the spins of the atoms of the medium causing a magnetisation per unit volume, M

$$\vec{B} = \mu_0\left(\vec{H} + \vec{M}\right) \qquad (1.30)$$

and

$$\vec{M} = \chi_M \vec{H} \qquad (1.31)$$

where χ_M is the **magnetic susceptibility**

Therefore in a medium

$$\vec{B} = \mu_0(1 + \chi_M)\vec{H} = \mu\mu_0\vec{H} \qquad (1.32)$$

Where μ is the relative magnetic permeability of the medium and

$$\mu = (1 + \chi_M) \qquad (1.33)$$

thus

$$\vec{H} = \frac{\vec{B}}{\mu\mu_0} \qquad (1.34)$$

The above analysis means the each medium has a value for the electric permittivity and for the magnetic permeability that differs from the values given above for vacuum. The permittivity which in free space was ε_0 is now $\varepsilon\varepsilon_0$ and the permeability in free space μ_0 becomes $\mu\mu_0$ within a medium. Non magnetic media have $\mu = 1$ and so the permeability remains unchanged, that is, there is no induced magnetisation per unit volume and $\chi_M = 0$.

We may summarise for the action of electric and magnetic fields on media;

Electric Field	**Magnetic Field**
The effect on the medium is ;	
Polarisation per unit volume	**Magnetisation per unit volume**
$\vec{P} = \varepsilon_0 \chi_E \vec{E}$	$\vec{M} = \chi_M \vec{H}$
We may introduce new field descriptions within the medium	
Electric Displacement	**Magnetic Field Induction**
$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$	$\vec{B} = \mu_0 (\vec{H} + \vec{M})$
Using the above	
$\vec{D} = \varepsilon_0 \vec{E}(1 + \chi_E)$	$\vec{B} = \mu_0 \vec{H}(1 + \chi_M)$
And a new material property is defined using the above	
Permittivity (Dielectric constant)	**Permeability**
$\vec{D} = \varepsilon\varepsilon_0 \vec{E}$	$\vec{B} = \mu\mu_0 \vec{H}$
$\varepsilon = (1 + \chi_E)$	$\mu = (1 + \chi_M)$

Thus, ***the Maxwell equations and everything that flows from them*** in the preceding section **may be modified by replacing ε_0 with $\varepsilon\varepsilon_0$ and by replacing μ_0 with $\mu\mu_0$** as we shall see when we deal with the wave equation in the next section.

In summary Maxwell's equations, 1.1b – 1.4b become:

$$\nabla \bullet \vec{E} = \frac{\rho}{\varepsilon\varepsilon_0} \qquad (1.1c)$$

$$\nabla \bullet \vec{B} = 0 \qquad (1.2c)$$

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt} \qquad (1.3c)$$

$$\nabla \times \vec{B} = \mu\mu_0 \vec{J} + \mu\mu_0\varepsilon\varepsilon_0 \frac{d\vec{E}}{dt} \qquad (1.4c)$$

We may use the newly defined ***D*** and ***H*** fields to write the Maxwell equations in an even more compact form recalling that we defined

$$\vec{D} = \varepsilon\varepsilon_0 \vec{E} \qquad \text{and} \qquad \vec{H} = \frac{\vec{B}}{\mu\mu_0}$$

$$\nabla \bullet \vec{D} = \rho \tag{1.1d}$$

$$\nabla \bullet \vec{B} = 0 \tag{1.2d}$$

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt} \tag{1.3d}$$

$$\nabla \times \vec{H} = \left(\vec{J} + \frac{d\vec{D}}{dt} \right) \tag{1.4d}$$

In the case of inhomogenous or anisotropic media such as crystals we will find that the situation regarding ε (or equivalently χ or n) is more complex and it is necessary to modify the dielectric constant (or equivalently the refractive index) to account for new phenomena arising from inhomogeneity or anisotropy.

(i) This will lead to the use of a tensor to describe the dielectric constant/refractive index when discussing birefringence and related phenomena.

(ii) Indeed, where the medium is absorbing or conducting it is the case that the dielectric constant will be complex, $\varepsilon = \varepsilon' + j\varepsilon''$, and the imaginary part related to the loss or conductivity, σ, by $\sigma = \omega\varepsilon_0\varepsilon''$.

We will deal with these complexities when they arise. For now the simple isotropic, homogenous medium approximation will be sufficient. Generally, apart from magnetic materials it is the case that $\mu = 1$. In particular, glasses, which are of great interest in optics, are generally non magnetic.

Electromagnetic Wave Equation in Simple Media

To obtain the wave equation in our simple medium we need to replace μ_0 with $\mu\mu_0$ and ε_0 with $\varepsilon\varepsilon_0$. The wave equation in our simple medium now becomes

$$\nabla^2 \vec{E} = \varepsilon\varepsilon_0 \mu\mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{v^2} \frac{\partial^2 \vec{E}}{\partial t^2} \tag{1.8a}$$

with a wave velocity in the medium of

$$v = \frac{1}{\sqrt{\mu\varepsilon\varepsilon_0\mu_0}} = \frac{c}{\sqrt{\mu\varepsilon}} = \frac{c}{n} \qquad (1.9a)$$

Here n is the refractive index and in non-magnetic media where $\mu \equiv 1$ we note

$$n = \sqrt{\varepsilon\mu} = \sqrt{\varepsilon} \qquad (1.35)$$

NB. ε (the relative permittivity) will vary with frequency but we ignore this for now and note that the relative permittivity and dielectric constant of a medium are one and the same.

It is worth noting here that $v = \frac{c}{n} = \frac{\omega}{k}$ in a medium of refractive index, n, and if an electromagnetic wave passes from free space into this medium, that is from a region where $v = c = \frac{\omega}{k}$ to a region (the medium) where $v = \frac{c}{n} = \frac{\omega}{k}$ this can either involve ω changing to $\frac{\omega}{n}$ or k changing to nk. It is the latter that changes with k becoming nk or λ becoming $\frac{\lambda}{n}$. The frequency of the wave does not alter. Generally I will denote the wavevector and wavelength in free space as k_0 and λ_0 respectively and in a medium of refractive index, n, we have a new wavevector and wavelength given by

$k = nk_0$ and $\lambda = \frac{\lambda_0}{n}$

The Relationship Between *E* and *H* in the Electromagnetic Field.

We can analyse further the plane wave solution by assuming that in free space (or our simple medium) the direction of the field vectors do not change as the wave propagates and also by assuming that ***B*** is described by the same plane wave (wavevector, angular frequency) as is ***E***. We may take the E field to be pointing in the x direction, ie E_x is the only component that exists and the plane wave propagates in the z direction (as usual).

Using equation 1.3a

$$\nabla \times E = -\frac{dB}{dt} \qquad \left(\frac{dE_z}{dy} - \frac{dE_y}{dz}\right), \left(\frac{dE_z}{dx} - \frac{dE_x}{dz}\right), \left(\frac{dE_y}{dx} - \frac{dE_x}{dy}\right) = -\frac{dB_x}{dt}, -\frac{dB_y}{dt}, -\frac{dB_z}{dt}$$

We can find two components of the *B* field, *y* and *z*, as the above does not allow an *x* component.

$$-\frac{dE_x}{dz} = -jkE_x = -\frac{dB_y}{dt} = j\omega B_y \qquad (1.36)$$

And recalling that for a plane wave propagating in the z direction the amplitude does not vary with x or y the above simplifies to

$$\frac{dE_x}{dy} = 0 = -\frac{dB_z}{dt} = j\omega B_z \qquad (1.37)$$

The *B* field is at right angles to the *E* field, pointing in the *y* direction and both *E* and *B* are at right angles to the direction of propagation.

Further we see that

$$\frac{E_x}{B_y} = \frac{\omega}{k} = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \qquad (138)$$

This is a very important relation between *E* and *B*.

The magnetic induction, *B* is sometimes dropped and $H = B/\mu_0$ is used instead. In this case

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \eta_0 = 377\Omega \qquad (1.39)$$

Where η_0 is the impedance of free space.

The impedance of a simple medium is then defined by analogy

$$\eta = \sqrt{\frac{\mu\mu_0}{\varepsilon\varepsilon_0}} \qquad (1.40)$$

Noting that for most media which are non-magnetic $\mu = 1$, 1.40 becomes

$$\eta = \frac{\eta_0}{\sqrt{\varepsilon}} = \frac{\eta_0}{n} \qquad (1.41)$$

and in a medium the relationship between E and H is

$$E = \eta H \tag{1.42}$$

Power Flow in the EM Wave & the Poynting Vector

If we wish to confer a physical reality upon the electromagnetic fields represented by our solutions to the electromagnetic wave equation we need to recognise that the electric and magnetic fields have an energy associated with them. This, we have seen in earlier electromagnetism courses where the energy in the electric field between two parallel plates separated by a distance d and with a potential difference V is given by

$$U = \frac{1}{2}QV \tag{1.43}$$

We see this by noting that the charging of the capacitor involved the movement of the charge Q through a potential difference V

Q is the charge stored on the capacitor is given by,

$$Q = CV = \frac{\varepsilon_0 A}{d}V \tag{1.44}$$

Where we assume a parallel plate capacitor with vacuum between two plates of area A separated by a distance d

$$U = \frac{1}{2}\frac{\varepsilon_0 A}{d}V^2 \tag{1.45}$$

This energy is stored in a volume $V = Ad$ by an electric field $E = \frac{V}{d}$ therefore the energy density associated with the electric field, u_E , is

$$u_E = \frac{U}{Ad} = \frac{1}{2}\varepsilon_0 E^2 \tag{1.46}$$

And we assume the same energy density associated with any electric field in free space.

Likewise for the magnetic field, we compute the energy associated with a current carrying toroid

$$u_B = \frac{1}{2}\frac{B^2}{\mu_0} \tag{1.47}$$

The electric and magnetic field energies exchange between one another as the electromagnetic field propagates and they are therefore equal

$$u_E = u_B \tag{1.48}$$

and the electromagnetic field energy density, u, is simply the sum of the two

$$u = u_E + u_B = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) = \varepsilon_0 E^2 = \frac{B^2}{\mu_0} \tag{1.49}$$

We are interested in knowing the amount of energy per unit time (power) crossing unit area, Wm^{-2}. To obtain this value we imagine a plane wave travelling in a direction perpendicular to an area *A*. In a time Δt a paralleliped of length $c\Delta t$ and of volume $c\Delta tA$ containing an energy $U = uc\Delta tA$ passes through the area *A*. We therefore find the energy per unit time per unit area or power per unit area, $\frac{P}{A}$ flowing across this area to be

$$\frac{P}{A} = S = \frac{U}{\Delta tA} = uc = c\varepsilon_0 E^2 = \frac{\varepsilon_0 E^2}{\sqrt{\varepsilon_0 \mu_0}} = \sqrt{\frac{\varepsilon_0}{\mu_0}}E^2 = \frac{E^2}{\eta_0} \tag{1.50}$$

Where we used the previously discovered relation between the velocity of light in free space, $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ and define $\sqrt{\frac{\mu_0}{\varepsilon_0}} = \eta_0 = 377\Omega$, as the electromagnetic impedance of free space. Or using the magnetic field

$$S = uc = c\frac{B^2}{\mu_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{\mu_0^2}B^2 = \eta_0 \frac{1}{\mu_0^2}B^2 = \frac{E^2}{\eta_0} \tag{1.51}$$

$$S = \frac{1}{\mu_0}EB \tag{1.52}$$

We make the reasonable assumption that the power is propagating in the same direction as the wave and write in vector notation

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} \tag{1.53}$$

And $\vec{S}$ or <u>***S***</u> is called the Poynting vector that shows the power per unit area or intensity in the electromagnetic field as it flows in a direction perpendicular to both <u>***E***</u> and <u>***B***</u> as defined by their cross product.

NB. Right hand rule: Index finger <u>E</u>, middle finger <u>B</u> and thumb <u>S</u>.

Finally we need to note that any instrument (including our eyes) that measure light intensity actually respond to the time averaged intensity and in this case the relation between light intensity (as measured) and the electric field is strictly given by

$$I = \frac{\langle E \rangle^2}{\eta_0} = \frac{E_0^2}{2\eta_0} \tag{1.54}$$

in free space where the triangular brackets indicate a time average which gives the 2 in the denominator for a sin or cosine time variation,

and in a medium of refractive index n or impedance $\eta = \sqrt{\frac{\mu\mu_0}{\varepsilon\varepsilon_0}}$

by

$$I = \frac{E_0^2}{2\eta} = \frac{nE_0^2}{2\eta_0} \tag{1.55}$$

Highlights of Part 1

The Electromagnetic Wave Equation	$\nabla^2 E = \varepsilon_0 \mu_0 \dfrac{\partial^2 E}{\partial t^2} = \dfrac{1}{c^2}\dfrac{\partial^2 E}{\partial t^2}$
The Velocity of Light	$c = \dfrac{1}{\sqrt{\varepsilon_0 \mu_0}} = 2.9979 \times 10^8 ms^{-1}$
The Phase Velocity in a Medium	$v = \dfrac{1}{\sqrt{\mu \varepsilon \varepsilon_0 \mu_0}} = \dfrac{c}{\sqrt{\mu \varepsilon}} = \dfrac{c}{n}$
The Refractive Index	$n = \sqrt{\varepsilon \mu} \approx \sqrt{\varepsilon}$
The Impedance of Free Space	$\dfrac{E_x}{H_y} = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = \eta_0 = 377\Omega$
The Impedance of a Medium	$\dfrac{E_x}{H_y} = \sqrt{\dfrac{\mu \mu_0}{\varepsilon \varepsilon_0}} = \eta$
For a Non Magnetic Medium	$\dfrac{E_x}{H_y} = \sqrt{\dfrac{\mu_0}{\varepsilon \varepsilon_0}} = \dfrac{\eta_0}{n}$
Relation between E and B in free space	$\dfrac{E_x}{B_y} = c$
Relation between E and B in a medium	$\dfrac{E_x}{B_y} = v = \dfrac{c}{n}$
Relation between E and Intensity in free space	$I = \dfrac{E_0^2}{2\eta_0}$
Relation between E and Intensity in a medium	$I = \dfrac{E_0^2}{2\eta} = \dfrac{nE_0^2}{2\eta_0}$