3. WAVEGUIDES

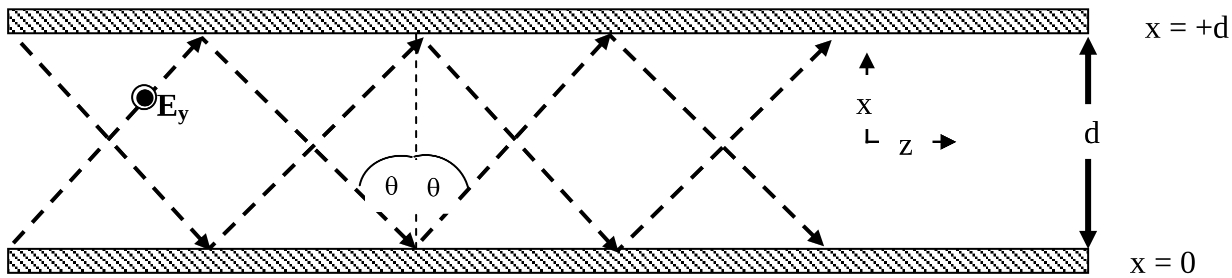
We move on in these notes to examine the possibility of guiding light in a desired direction by its confinement in a designed structure. The first means that springs to mind is to use the phenomenon of total internal reflection that was discovered in the previous section where it is recalled, light sourced in a medium of higher refractive index, n_S , will, at the boundary with a dielectric of lower refractive index, n_U , undergo total internal reflection if the angle of incidence at the interface is above the critical angle, $\theta_C = sin^{-1}\left(\frac{n_U}{n_S}\right)$. To do this a symmetric, dielectric slab waveguide is created with a high refractive index, dielectric slab, known as the guide, bounded on two sides by dielectrics of equal and lower refractive index known as the cladding dielectric.

Such a system while presenting no great difficulties in analysis is however a less straightforward system than that of guiding electromagnetic waves using reflection off of a metal surface and thence guiding the wave between two metal plates. It is therefore this system that we begin by examining.

Metal Guides.

The simplicity of metal waveguides lies in the fact that at the metal/air interface there is no evanescent field into the metal and therefore no phase change to take account of. The boundary condition at the metal/air interface is simply that the electric field must be zero. This is because ideal metals (perfect conductors) do not support an electric field. Any attempt to introduce an electric field in a metal will result in the re-arrangement of the copious mobile electrons available in the metal in such a way as to polarise the metal thus negating the electric field and to do this quickly! Nevertheless an analysis of the metal guide is very useful as it will bring out many of the features that are found in a dielectric slab waveguide but in a simpler fashion, and therefore will serve as a useful introduction. The simplified system and the concept of modes may be introduced by considering the guiding of an electromagnetic wave between two parallel metal plates, which are perfect reflectors (infinite conductivity), separated by a distance d,. This is not achieved in practice but at microwave frequencies it is approximated sufficiently well. For optical frequencies the losses are too great and the dielectric slab waveguide is used but analysis of the simpler system is a useful starting point.

The system analysed in the following is illustrated below and the axes defined where we assume the wave to be guided in the z direction and bound in the x direction with E field polarised in the y direction (a TE wave).



The field will be a superposition of "upward" and "downward" travelling waves, making an angle θ to the x axis, either of which is of the form

$$E_y = E_{\pm}\, exp\left(-\, jk_0 \left\{ z\, sin\,\theta \pm x\, cos\,\theta \right\}\right) \tag{3.1}$$

Where examination of the diagram has shown

$$k_x = \pm k_0\, cos\,\theta \qquad \text{and} \qquad k_z = k_0\, sin\,\theta \tag{3.2}$$

At either of the metal surfaces superposition will then be a sum of a downward travelling wave (eg. incident) and an upward travelling wave (eg. reflected) and the resultant field is given by

$$E_y = E_+ \exp\left(-\, jk_0 \left\{ z \sin\theta - x\cos\theta \right\}\right) + E_- \exp\left(-\, jk_0 \left\{ z \sin\theta + x\cos\theta \right\}\right) \tag{3.3}$$

The signs of the $x\cos\theta$ terms in the above represent the fact that the downward travelling field is travelling in the –x direction and vice versa

The boundary conditions must be satisfied at $x = 0$ and $x = +d$. These are that E vanishes at these values of x as the metals are perfect conductors. The first condition for $x = 0$ is satisfied if $E_+ = -E_- = E_0$.

and thus

$$E_y = E_0 \exp(-jk_0 z \sin\theta)[\exp(+jk_0 x \cos\theta) - \exp(-jk_0 x \cos\theta)]$$

We may use De Moivres theorem to rewrite the exponential in the square bracket and obtain

$$E_y = 2E_0 \sin(k_0 x \cos\theta) \exp(-jk_0 z \sin\theta) \tag{3.4}$$

This solution is that of a wave travelling in the z direction with a sinusoidal amplitude envelope in the x direction. ie

$$E_y = E(x) \exp(-j\beta z) \tag{3.5}$$

where β is the called the propagation coefficient and is the wave vector in the direction of propagation (in this case z). $\beta = k_0 \sin\theta$ and $E(x) = 2E_0 \sin(k_0 x \cos\theta)$

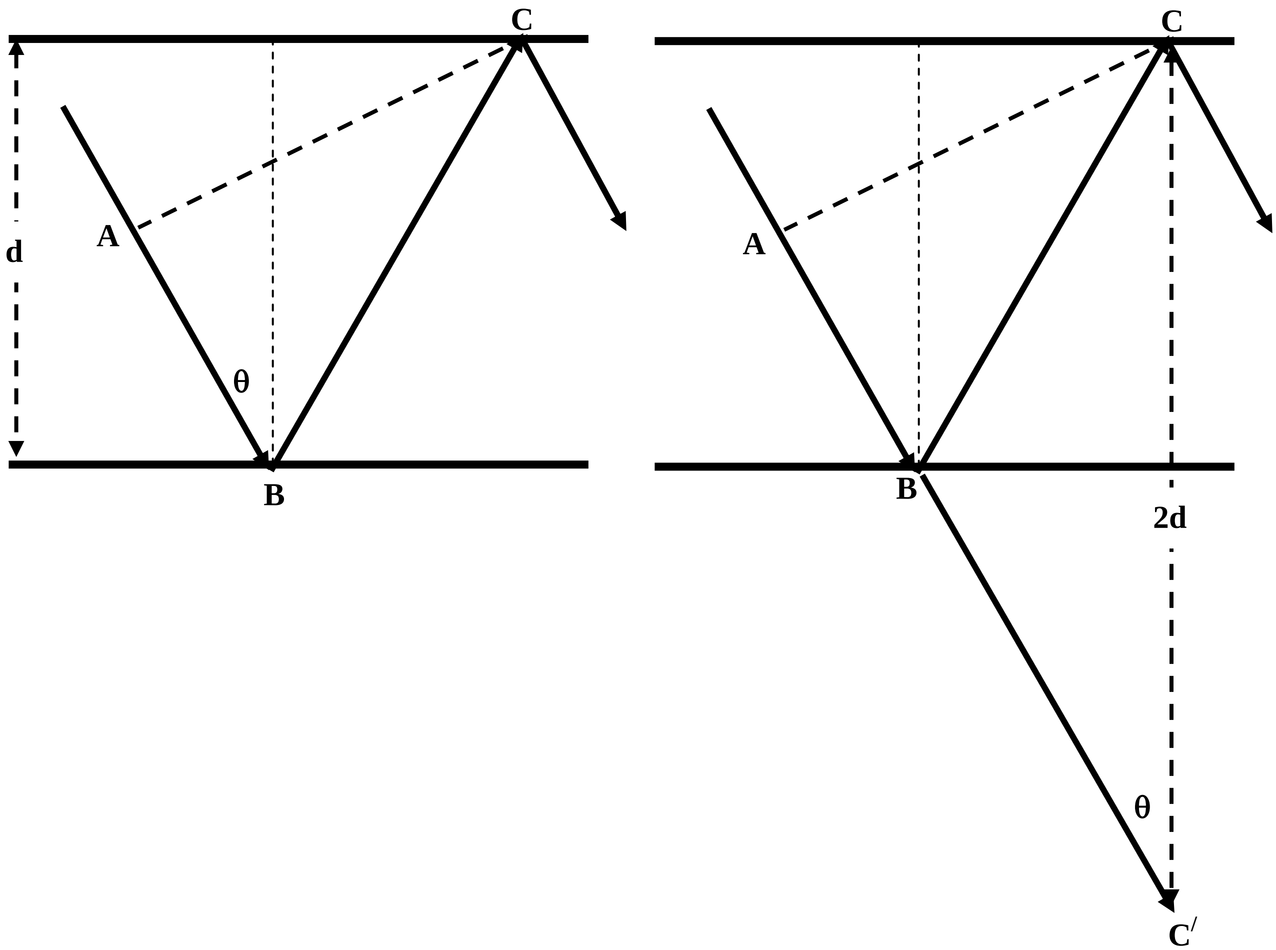
The second condition requires

$$2 \sin(k_0 d \cos\theta) = 0 \tag{3.6}$$

Or

$$k_0 d \cos\theta = m\pi, \qquad m = 1,2,3... \tag{3.7}$$

Equ 3.7 is the eigenvalue equation of the guide and is the condition that sets the modes of the guide. This is the transverse resonance condition and may be understood with reference to the following diagram;



A guided ray is shown in the left hand figure and a constant phase front perpendicular to the direction of propagation of the ray by the dashed line. Concentrating on the path ABC it is necessary that over the path ABC the ray accumulates a change in the phase of the wave equal to an integral multiple of 2π as both A and C lie on the same constant phase front. ie the phase change is ,

$$\Delta\phi = k_0 ABC = m2\pi$$

Each value of m corresponds to a particular guided mode. The length *ABC* can be calculated by the construction in the figure on the right which “unfolds the ray" and we have $ABC = 2d\cos\theta$ hence the condition, $2k_0 d\cos\theta = 2m\pi$.

Consideration of (3.7) leads to a number of conclusions:

1) If $\frac{m\pi}{k_o d} > 1$ there is no solution

2) So for any guided mode at all to exist

$$d > \frac{m\pi}{k_0} \tag{3.8}$$

3) In other words if d is fixed there is a value of k_0

$$k_0 = \frac{m\pi}{d} \tag{3.9}$$

below which no modes exist.

4) Writing this in terms of frequency, there exists a cut off frequency,

$$\nu_{CO} = \frac{ck_0}{2\pi} \tag{3.10}$$

below which no modes exist.

The guide is seen to act as a high pass filter. This is why daylight can be seen when looking down a section of microwave guide!

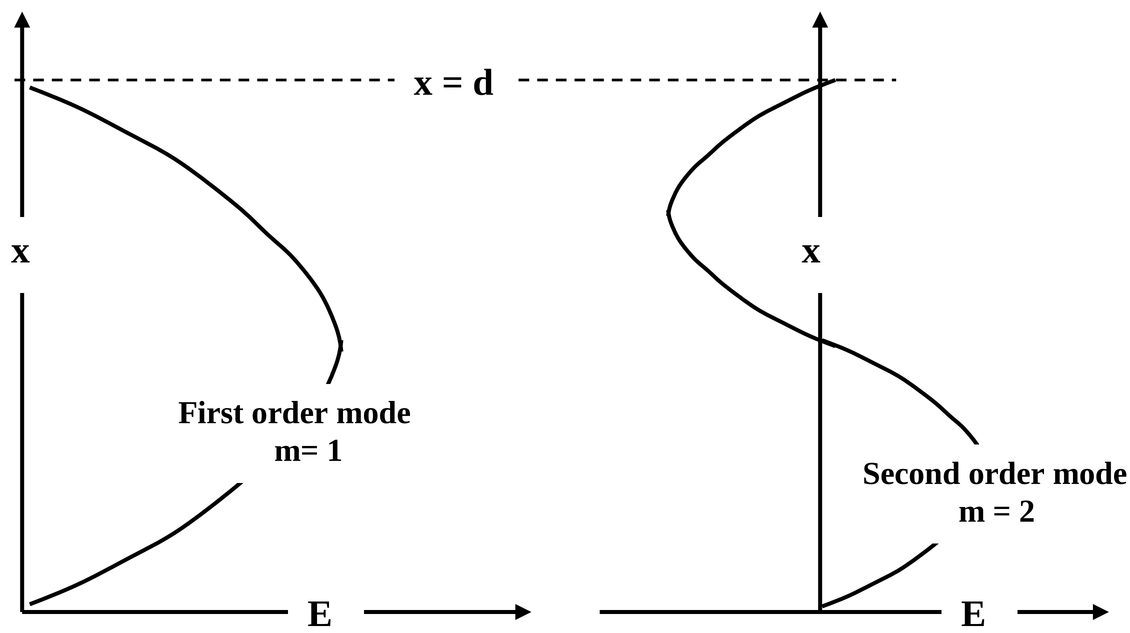
5) Alternatively no modes are supported if the guide is too small and 3.9 may be recouched

$$d < \frac{\lambda}{2} \tag{3.11}$$

6) If d is just slightly bigger than $\frac{\lambda}{2}$ then only one mode is supported. Larger values of d lead to multimoded guides with some low frequency cutoff. The larger m (the higher the order of the mode) the smaller θ will be and at cutoff $\theta = 0$ at which point the waves just bounce up and down between the walls without propagation down the guide.

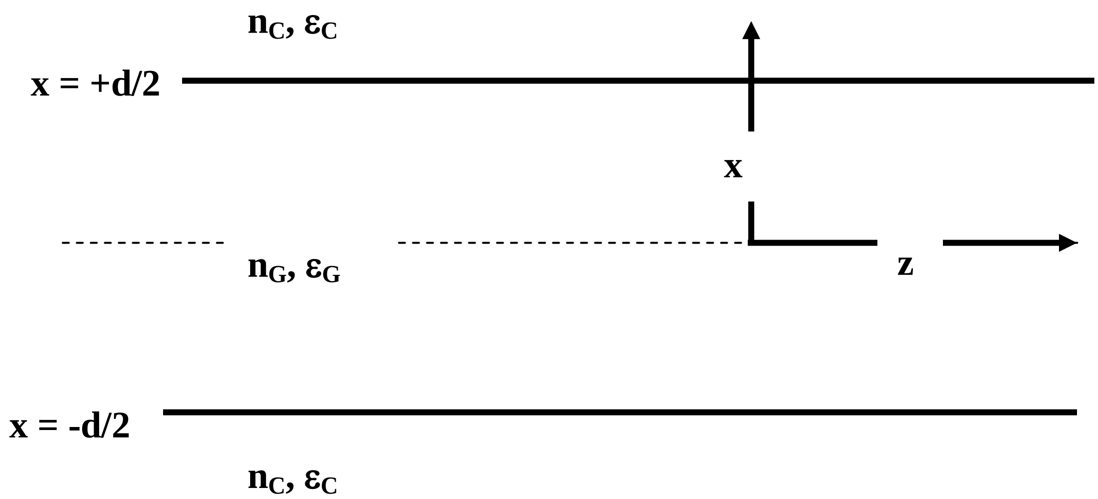
The analysis of this simpler system has demonstrated many of the important features of the dielectric slab waveguide whose analysis is more difficult due to the change of phase on reflection which is itself a function of θ.

The first and second order modes of the metallic guide are sketched below.



The Symmetric Slab Waveguide

Physical Structure



The simple structure outlined above will act as a waveguide, if $n_G > n_C$, for waves travelling at an angle wrt the normal to the interface greater that the critical angle. It is however the case that not any and all angles greater than the critical angle will correspond to a guided wave and the possible guided electromagnetic fields are limited as modes (and superpositions of modes) of the guide. We have seen this in the preceding analysis of a metal waveguide. The following notes use the fact that the boundary conditions at a dielectric interface require that the tangential ***E*** and ***H*** fields are continuous across the boundaries involved in order to analyse this problem. For simplicity we will look at the transverse electric, TE, mode with its polarisation in the y direction and therefore completely tangential to the interface. With E_y there is an orthogonal ***H*** field with x and *z* components whose z component will be the tangential component.

Using Maxwells curl equation for ***E***

$$\nabla \times E = -\mu \frac{\partial H}{\partial t} = j\omega\mu H \tag{3.12}$$

where a time dependence for $\underline{\boldsymbol{H}}$ of the form $\underline{\boldsymbol{H}} = \underline{\boldsymbol{H}}_0 \exp(j\omega t)$ has been assumed to obtain the time derivative. We find for H_z only one component of the curl

$$H_z = \frac{1}{j\omega\mu}\left(\frac{\partial}{\partial x}E_y(x,z) - \frac{\partial}{\partial y}E_x(x,z)\right) = \frac{1}{j\omega\mu}\frac{\partial}{\partial x}E_y(x,z) \qquad (3.13)$$

a) Field variation in the x direction within the cladding.

We have seen previously that due to the requirement that k_z (= β) matches at the two interfaces, k_x in regions 1 and 3 must be entirely imaginary for guided modes ($\theta > \theta_C$).

$$k_{Cx} = j\sqrt{\beta^2 - k_C^2} = j\sqrt{k_G^2 \sin^2\theta - k_C^2} = jk_0\sqrt{n_G^2 \sin^2\theta - n_C^2} = \pm j\alpha \qquad (3.14)$$

and the fields are therefore of the form $A_1\exp(-\alpha x)$ in cladding region 1 and $A_1\exp(\alpha x)$ in cladding region 3. This is the evanescent field just outside the guide exponentially decaying over a length scale represented by α^{-1}.

b) Field variation in the x direction within the guide.

Within the guide there are positive and negative travelling waves in the x direction giving rise to standing waves described by cosinusoidal functions, $A_2\cos(k_{Gx}x)$ for symmetric modes (even parity) and $A_2\sin(k_{Gx}x)$ for anti-symmetric modes (odd parity).

$$k_{Gx} = \sqrt{k_2^2 - \beta^2} \qquad (3.15)$$

c) Field variation in the propagation direction , z.

Within the guide the fields will propagate as $\exp(j\beta z)$ in the z direction everywhere and from now on **we write k_z as β, the propagation constant.**

Electric Fields in Guide

The electric fields may then be written, following the above prescriptions for the three regions, as follows

$$\mathbf{E_y(x,z)} = \left.\begin{array}{ll} A_1\exp(-\alpha x) & \\ A_2\begin{pmatrix}\cos(k_{Gx}x) \\ \\ \sin(k_{Gx}x)\end{pmatrix} & \\ \pm A_1\exp(\alpha x) & \end{array}\right\} \exp(j\beta z) \qquad \begin{array}{l} x > d/2 \\ \\ d/2 < x < -d/2 \\ \\ x < -d/2 \end{array} \qquad (3.16)$$

NB the choice of a cosine or sine variation for the field within the guide represents the possibility of even/odd parity (symmetric/anti-symmetric) modes respectively.

Magnetic Fields in Guide

The tangential magnetic fields are found by using $H_z = \frac{1}{j\omega\mu}\frac{\partial}{\partial x}E_y(x,z)$ and may be written, following the above prescriptions for the three regions and taking the derivative wrt x of the electric fields given above, as follows

$$\boldsymbol{H_z(x,z)} = \frac{1}{j\omega\mu} \times \left\{\begin{array}{ll} -\alpha A_1\exp(-\alpha x) & \\ \mp k_{Gx}A_2\begin{pmatrix}\sin(k_{Gx}x) \\ \\ \cos(k_{Gx}x)\end{pmatrix} & \\ \alpha A_1\exp(-\alpha x) & \end{array}\right\} \exp(j\beta z) \qquad \begin{array}{l} x > d/2 \\ \\ d/2 < x < -d/2 \\ \\ x < -d/2 \end{array} \qquad (3.17)$$

Note that the even/odd parity electric fields of 3.16 are represented by the sine/cosine variation in the magnetic fields in 3.17 after the differentiation wrt *x*.

Guidance Conditions

To obtain the guidance conditions for the guide, **in order to find out which values of β represent a valid guided mode** for the particular guide under consideration, both E_y and H_z must be matched at $x = +d/2$ and $-d/2$

Symmetric guided modes.

For symmetric E fields the field matching at $x = +\frac{d}{2}$ gives

$$A_2 \cos\left(\frac{k_{Gx} d}{2}\right) = A_1 \exp\left(\frac{-\alpha d}{2}\right) \tag{3.18a}$$

For symmetric H fields the field matching at $+\frac{d}{2}$ gives

$$-k_{Gx} A_2 \sin\left(\frac{k_{Gx} d}{2}\right) = -\alpha A_1 \exp\left(\frac{-\alpha d}{2}\right) \tag{3.18b}$$

Dividing equation 3.18a by 3.18b and re-arranging we have

$$\frac{\alpha}{k_{Gx}} = \tan\left(\frac{k_{Gx} d}{2}\right) \tag{3.18c}$$

This is the guidance condition for symmetric modes of the guide in its basic form. We shall simplify it in order to make it more tractable after obtaining the guidance condition for anti-symmetric modes of the guide.

Anti-symmetric guided modes

For anti-symmetric E fields matched at $x = +\frac{d}{2}$

$$A_2 \sin\left(\frac{k_{Gx} d}{2}\right) = A_1 \exp\left(\frac{-\alpha d}{2}\right) \qquad (3.19a)$$

For anti-symmetric H fields matched at $x = +\frac{d}{2}$

$$-k_{Gx} A_2 \cos\left(\frac{k_{Gx} d}{2}\right) = \alpha A_1 \exp\left(\frac{-\alpha d}{2}\right) \qquad (3.19b)$$

Dividing (3.20a) by (3.20b) and re-arranging we have

$$\frac{\alpha}{k_{Gx}} = -\cot\left(\frac{k_{Gx} d}{2}\right) \qquad (3.19c)$$

This is the guidance condition for anti-symmetric modes of the guide in its basic form.

We now seek to write these two sets of conditions, 3.18c and 3.19c, in a simpler form in order that a graphical representation of the conditions may be easily obtained. To do this we first need to define normalised transverse wavevectors.

NB we have no need to apply the boundary condition at the $-\frac{d}{2}$ boundary as this would bring no new information. This is because the guide is symmetric and we go from $-\frac{d}{2}$ to $+\frac{d}{2}$ by inverting the symmetric guide with no actual physical change in the problem being considered. Anti-symmetric guides also exist where $n_{C1} \neq n_{C3}$ and $n_G > n_{C1}$, n_{C3}. In that case both boundary conditions would be used. This does not concern us here.

Normalised Transverse Wavevectors, *U*, *W* and *V* and Characteristic Equation of Slab Guide

Another way to represent the guidance conditions graphically is to rewrite them multiplying both sides by $\frac{d}{2}$ as follows;

The condition for symmetric modes

$$\frac{\alpha}{k_{Gx}} = tan\left(\frac{k_{Gx}d}{2}\right) \tag{3.18c}$$

becomes

$$\frac{\alpha d}{2} = \frac{k_{Gx}d}{2} tan\left(\frac{k_{Gx}d}{2}\right) \tag{3.20a}$$

And the condition for antisymmetric modes

$$\frac{\alpha}{k_{Gx}} = -cot\left(\frac{k_{Gx}d}{2}\right) \tag{3.19c}$$

becomes

$$\frac{\alpha d}{2} = -\frac{k_{Gx}d}{2} cot\left(\frac{k_{Gx}d}{2}\right) \tag{3.20b}$$

These new formulations of the guidance conditions lead us to define some new parameters of the waveguide, namely the normalised transverse wavevectors.

Normalised Transverse Wavevectors

With the new formulation for the guidance conditions equs 3.20, it becomes useful to define **normalised transverse wavevectors** in cladding and guide regions as follows:

(i) By transverse we indicate that it is the wavevector transverse to the guide direction, ie. k_x with which we are concerned

(ii) We normalise them (make them dimensionless) by multiplying by a natural length of the guide, $\frac{d}{2}$, to give:

a) The **normalised exterior transverse wavevector** , W , defined as

$$W = jk_{Cx}\frac{d}{2} = \frac{\alpha d}{2} = \left(\sqrt{\beta^2 - k_C^2}\right)\frac{d}{2} \tag{3.21a}$$

b) The **normalised interior transverse wavevector**, U defined as

$$U = k_{Gx}\frac{d}{2} = \left(\sqrt{k_G^2 - \beta^2}\right)\frac{d}{2} \tag{3.21b}$$

The guidance equations may then be written

$$W = U\tan U \tag{3.22a}$$

and

$$W = -U\cot U \tag{3.22b}$$

We can square and add equations 3.21a and 3.21b to give

$$U^2 + W^2 = \left(k_G^2 - k_C^2\right)\left(\frac{d}{2}\right)^2 = (n_G^2 - n_C^2)k_0^2\left(\frac{d}{2}\right)^2 = V^2 \qquad (3.23)$$

where V is an extremely important parameter in the description of waveguides and fibres known as the **normalised frequency**.

The equation

$$U^2 + W^2 = V^2 \qquad (3.23)$$

is known as **the characteristic equation of the guide** (or step index fibre). It is an easy equation to examine graphically.

To examine the guidance conditions graphically we plot a graph of W vs U and on this graph we plot

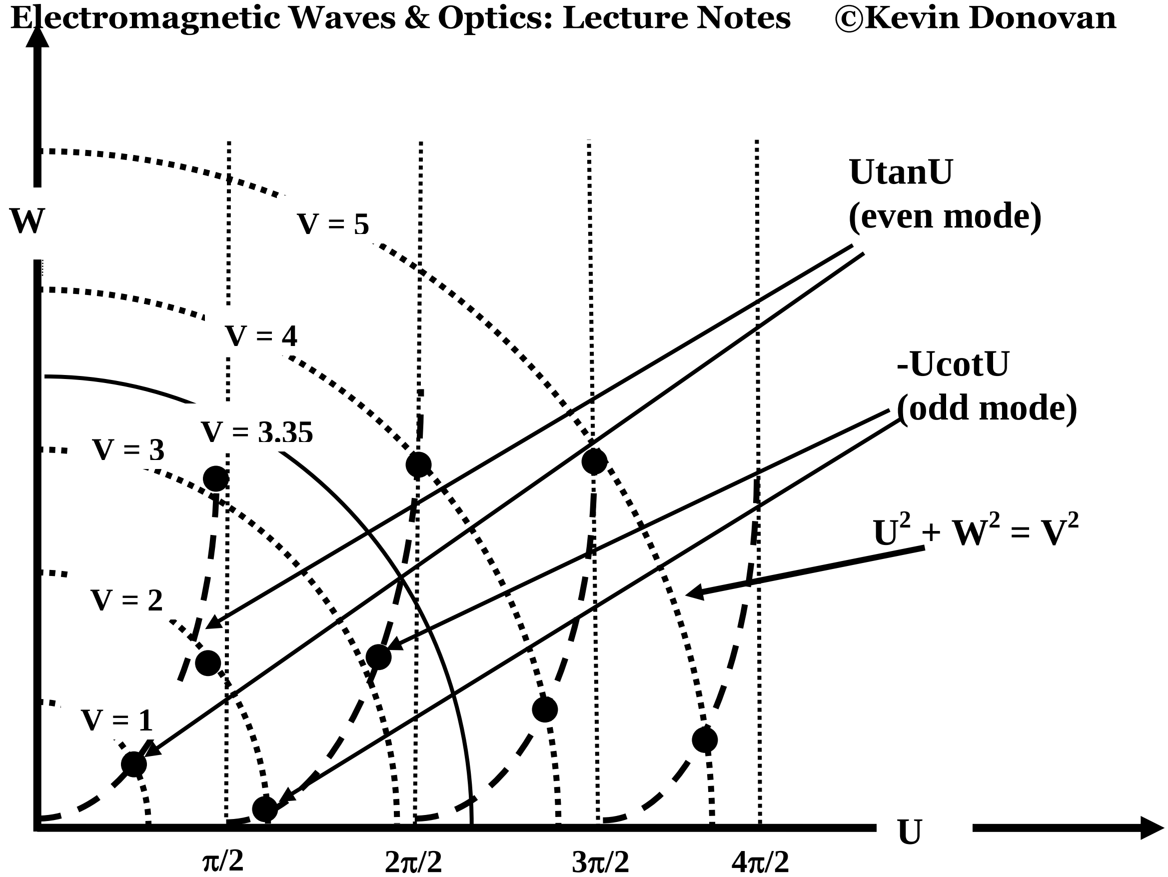
$W = U \tan U$ even modes

$W = -U \cot U$ odd modes

along with the characteristic equation rewritten as

$$W^2 + U^2 = V^2.$$

This is the equation of a circle of radius V shown schematically on the next page along with the guidance conditions.



Graphical solution of guidance conditions

Where the $U\tan U$ (or $-U\cot U$ curve) and $W=\sqrt{V^2-U^2}$ curves (quarter circles) cross one another the values of W and U represent those for an allowed mode at that frequency, V, ie an allowed mode of the particular guide. Remember that V represents a particular value of frequency for a given guide (defined through n_C , n_G and $\frac{d}{2}$).

Single Mode Operation.

From the graph we can see that as V is increased (the quarter circles have radius V), each time it increases by $\frac{\pi}{2}$ another TE mode is allowed (there is an equivalent diagram for TM modes) and therefore the number of TE modes propagating, N_P will be given by

$$N_P=\left(\frac{V}{\pi/2}+1\right)\approx\frac{2V}{\pi}. \tag{3.24a}$$

Including TM modes then we have a total of

$$N_P = \frac{4V}{\pi} + 2 \tag{3.24b}$$

for a multimode guide.

Generally the spatial distribution of light across the guide will be describable as a superposition of modes. It is desirable in many circumstances to allow only one mode to propagate. Notably, because phase and group velocities differ for different modes and if we desire to transmit a light pulse we reduce the temporal spreading of the pulse by allowing only one mode to propagate. From the graphical solution in the diagram, if V is less than π/2 only the first even mode will propagate. Ie when the condition

$$V = k_0 \sqrt{n_G^2 - n_C^2}\,\frac{d}{2} < \pi / 2 \tag{3.25}$$

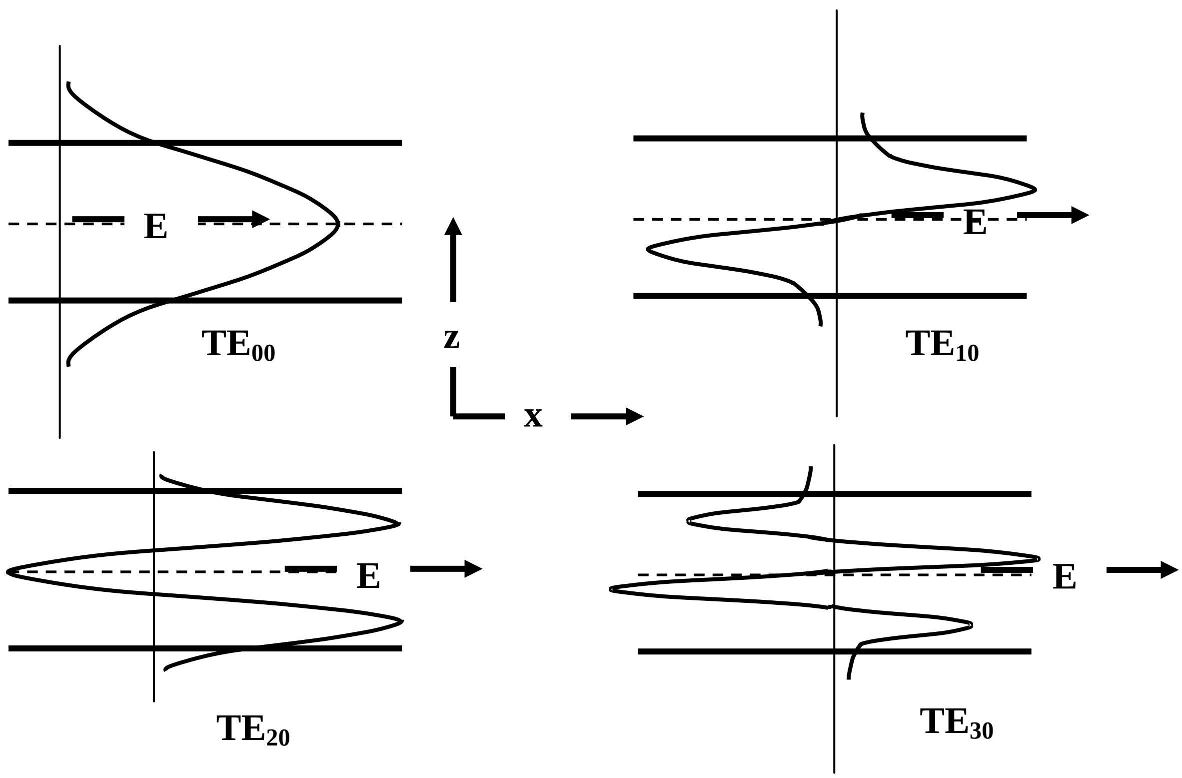
is satisfied only one mode will propagate. Thus for a given frequency $\omega = ck_0$ we will have **single mode propagation when**

$$d_{SM} < \frac{\pi c}{\omega\sqrt{n_G^2 - n_C^2}} \tag{3.26}$$

$$\omega_{CO} = \frac{\pi c}{d\sqrt{n_G^2 - n_C^2}} \qquad (3.27$$

Modal Behaviour

The crossing points of the two curves in the above figure represent allowed modes. If the *U*tan*U* curve is crossed an even, symmetric, mode ($E = A cos(k_{Gx} x)$) is represented and an odd, anti-symmetric, mode ($E = A sin(k_{Gx} x)$) if the –*U*cot*U* curve is crossed. The modes describe the distribution of the electric field in the x direction across the guide with exponentially decaying fields into the cladding.



The above diagrams show the variation of E_y with x for the lowest four TE modes, two even and two odd. These will be the same as the variation of H_y with x for the lowest four TM modes.

NB the mode is described by two suffices, TE_{mn} , and the two suffices are to describe the field variation in the *x* and the *y* direction. We have assumed the extent of the guiding and cladding regions to be infinite in the y direction in order to simplify the problem. An actual guide would have a more complicated structure in order to contain the electromagnetic wave in the *x* and *y* dimensions. Specification of a mode would then require the two mode numbers *m* and *n* to be given. To find the fields and allowed modes for this problem is beyond the scope of the course.

We are now in a position to examine more closely the modal behaviour in two limits.

From the graphical solution of the guidance conditions shown earlier we see that the value of U for the p^{th} mode lies between limits given by

$$(p-1)\frac{\pi}{2} < U_P < p\frac{\pi}{2} \Rightarrow (p-1)\frac{\pi}{2} < \left((k_{Gx})_P \frac{d}{2}\right) < p\frac{\pi}{2} \qquad p = 1, 2, 3 \qquad (3.28)$$

For odd p this represents a symmetric mode and for even p an anti-symmetric mode.

Limit 1. Low Frequency (Near Cutoff)

Looking at the graphical solution we see that the pth mode tends to cut off as U_P and V approach $(p-1)\frac{\pi}{2}$ and W approaches 0.

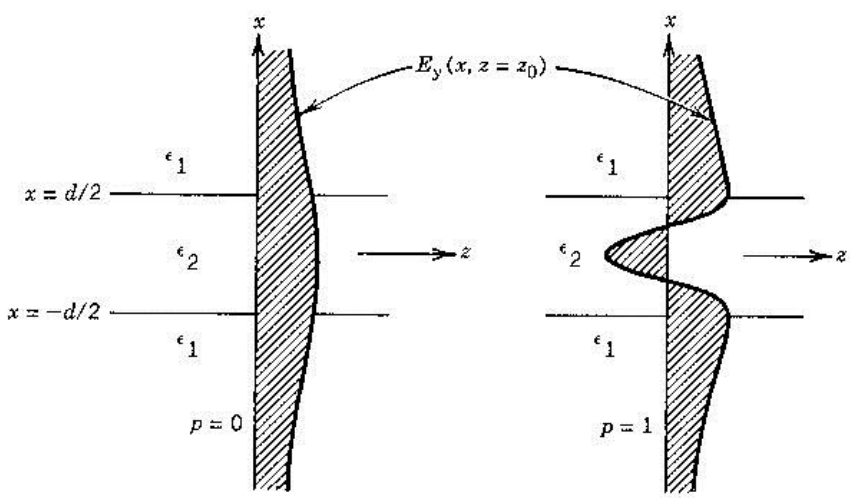
From the characteristic equation, $U^2 + W^2 = V^2$, we see that as this limit $U \to V$ is approached then we also require that $W \to 0$.

Recalling the definition of W (the exterior normalised transverse wavevector)

$$W = \alpha\frac{d}{2} \to 0 \qquad \Rightarrow \qquad \alpha \to 0 \qquad (3.21a)$$

The significance of this limit is realised where we know that the evanescent fields extend beyond the guide a characteristic distance α^{-1} and we see that this field is extending further for the p^{th} mode as the cutoff condition is approached and $\alpha \to 0$.

The first two even modes are illustrated below in this limit.



The first two even modes near to cutoff with extensive leakage into cladding region

From the graphical solution of the guidance equations it is also seen that as this limit is approached and k_{Gx} is reduced, recalling the relation between k_{Gx} and frequency;

$$k_{Gx} = n_G k_0 \cos\theta = \frac{\omega}{c} n_G \cos\theta$$

the frequency of the guided mode (or *V*) is also being reduced and there exists a frequency at which the mode is lost, referred to as the cut-off frequency for that mode. This cut-off frequency may be written as;

$$U_{P_{CO}} = V_{P_{CO}} = p\frac{\pi}{2} \tag{3.29}$$

And

$$V_{P_{CO}} = \left(\sqrt{n_G^2 - n_C^2}\right) k_{0_{CO}} \frac{d}{2} = p\frac{\pi}{2} \tag{3.30}$$

$$k_{0CO}\left(\sqrt{n_G^2 - n_C^2}\right) = \frac{\omega_{CO}}{c}\left(\sqrt{n_G^2 - n_C^2}\right) \rightarrow \frac{p\pi}{d} \tag{3.31}$$

or

$$\nu_{CO} = \frac{\omega_{CO}}{2\pi} = p\frac{c}{2d}\frac{1}{\sqrt{n_G^2 - n_C^2}} \tag{3.32}$$

Note that the $p = 0$ mode, **the zeroth order symmetric mode has no cutoff and will propagate at all frequencies down to zero**.

Limit 2. High Frequency (Far from Cutoff)

We now look at the other, high frequency limit for the *p*th mode.

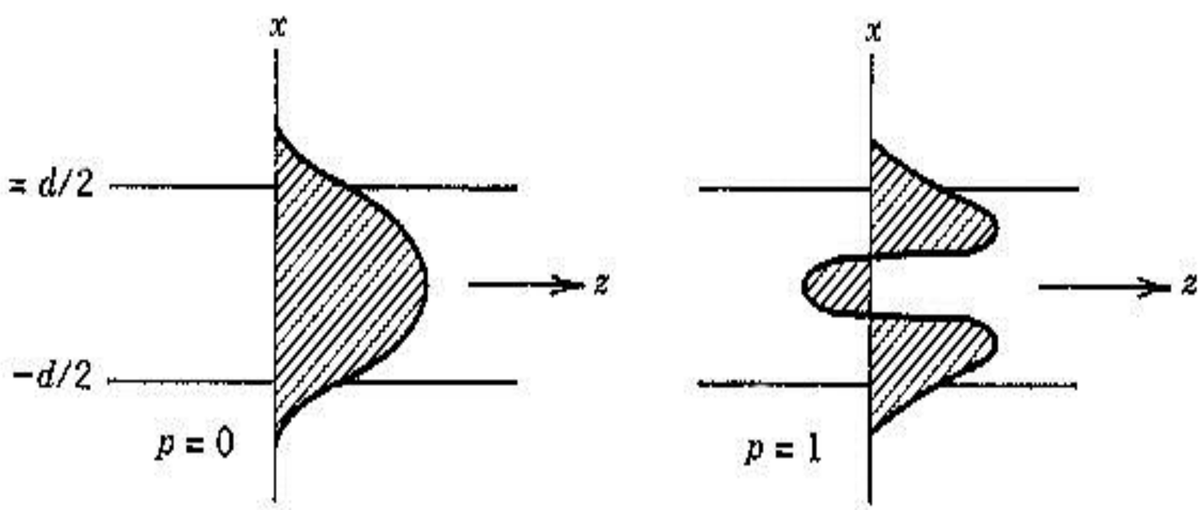
$$U_P = \left((k_{2x})_P \frac{d}{2} \right) \rightarrow p\frac{\pi}{2} \tag{3.33}$$

As from the guidance conditions $W_P = U_P \tan U_P$ and this tends asymptotically to infinity at this limit

$$W_P = \alpha_P \frac{d}{2} \rightarrow \infty \tag{3.34}$$

As $\alpha_P \rightarrow \infty$ the evanescent field in the cladding becomes more and more closely confined to the interface ($\alpha_P^{-1} \rightarrow 0$)

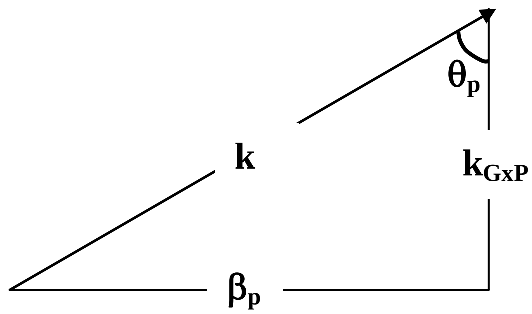
The first two even modes at this limit (far from cutoff) and tightly confined to guiding region are shown below.



We can examine the high/low frequency limits within a geometric ray representation. The p^{th} mode propagating at a given frequency in a given guide (ie dimension *d*, n_C and n_G)

will have a unique propagation constant β_P and k_{GxP} and it will travel at some angle θ_P such that

$$tan\,\theta_p = \frac{\beta_p}{(k_{Gx})_p} \tag{3.35}$$



The angle at which mode p propagates is frequency dependent and will be determined from the guidance condition.

Propagating in a given mode, p, and increasing frequency (or equivalently with the graphical solutions in mind, V) we have seen that U_P or k_{GxP} tend to asymptotic limits

$$U_P = \left((k_{Gx})_P \frac{d}{2} \right) \rightarrow (p+1)\frac{\pi}{2} \qquad \text{as } V \text{ is increased}$$

whereas β carries on increasing and approaches k_G therefore as ω is increased $\tan\theta_P$ increases more rapidly and θ_P approaches $\frac{\pi}{2}$. ie at higher frequencies the mode is travelling at increasingly grazing angles to the interface.

As the frequency (or V) is reduced toward cutoff, α approaches zero and from the graphical solution we see that;

$$U_P = k_{Gx}\frac{d}{2} \rightarrow V \qquad \Rightarrow \qquad k_{Gx} \rightarrow V\frac{2}{d} = \sqrt{n_G^2 - n_C^2}\,k_0$$

and

$$\beta_P \rightarrow k_C = n_C k_0$$

Therefore in this limit we have

$$tan\,\theta_P = \frac{\beta_P}{(k_{Gx})_P} \rightarrow \frac{k_C}{\sqrt{n_G^2 - n_C^2}\,k_0} = \frac{n_C}{\sqrt{n_G^2 - n_C^2}} \tag{3.36}$$

alternatively written (form the right angle!)

$$sin\,\theta_P \rightarrow \frac{n_C}{n_G} \tag{3.37}$$

and $\theta_P \rightarrow \theta_C$ the critical angle.

This is as we expect of course i.e. that as cutoff is approached for a given mode and guide by lowering the frequency the ray angle decreases until at cutoff it is equal to the critical angle. Any further decrease and no guiding occurs.

Dispersion relations (ω vs β)

A major interest in waveguides is of course their use in transmitting information in the form of streams of light pulses. Of great importance in this application is the propensity of the individual pulses to spread in time as they propagate, a tendency which it is necessary to avoid in order that information is not degraded or lost altogether. The degree to which this occurs is termed dispersion and will depend on several possible mechanisms which we shall look at presently. For now it is sufficient to note that a pulse is made up of a Fourier superposition of cosinusoidal waves and to understand the dispersion we need to look at the phase and group velocities of the waves in any particular mode. To find the phase and group velocities of a mode it is necessary to know the relationship between ω and β, $\omega(\beta)$. We can find from the graphical solution a value of $(k_{Gx})_P$ for any given k_G and thus find β_P. Each mode will have a different dispersion relation, $\omega_P(\beta_P)$. and in general to find this will involve numerical or graphical solution. We may however look at our two limiting situations again.

Limit 1. Low Frequency (Near Cutoff)

From the equation $\alpha^2 = \beta^2 - k_C^2$ and with $\alpha \to 0$ near cutoff as we saw earlier,

$$\beta^2 \to k_C^2 = \left(\frac{\omega}{c/n_C} \right)^2 \tag{3.38}$$

Giving the dispersion relation in the far from cutoff limit;

$$\omega = \frac{c}{n_C}\beta \tag{3.39a}$$

The mode number, *p* doesn't appear in the dispersion relation in this limit, thus, the dispersion relation is the same for all modes and the phase velocity of the modes approaches $\frac{c}{n_C}$, in the limit far from cutoff ie. it tends to the velocity of light in the cladding. We can easily understand why this should be the velocity of light in the cladding as the field extends a long way into the cladding when $\alpha \to 0$ and a larger fraction of the modal power is contained in the cladding regions.

Limit 2. High Frequency (Far from Cutoff)

For sufficiently large frequency we have already seen that the transverse wavevector k_{Gx} approaches

$$(k_{Gx})_P \to p\frac{\pi}{d}$$

We also have seen in this limit that

$$\beta \to k_G = \frac{\omega}{c/n_G} = n_G \frac{\omega}{c}$$

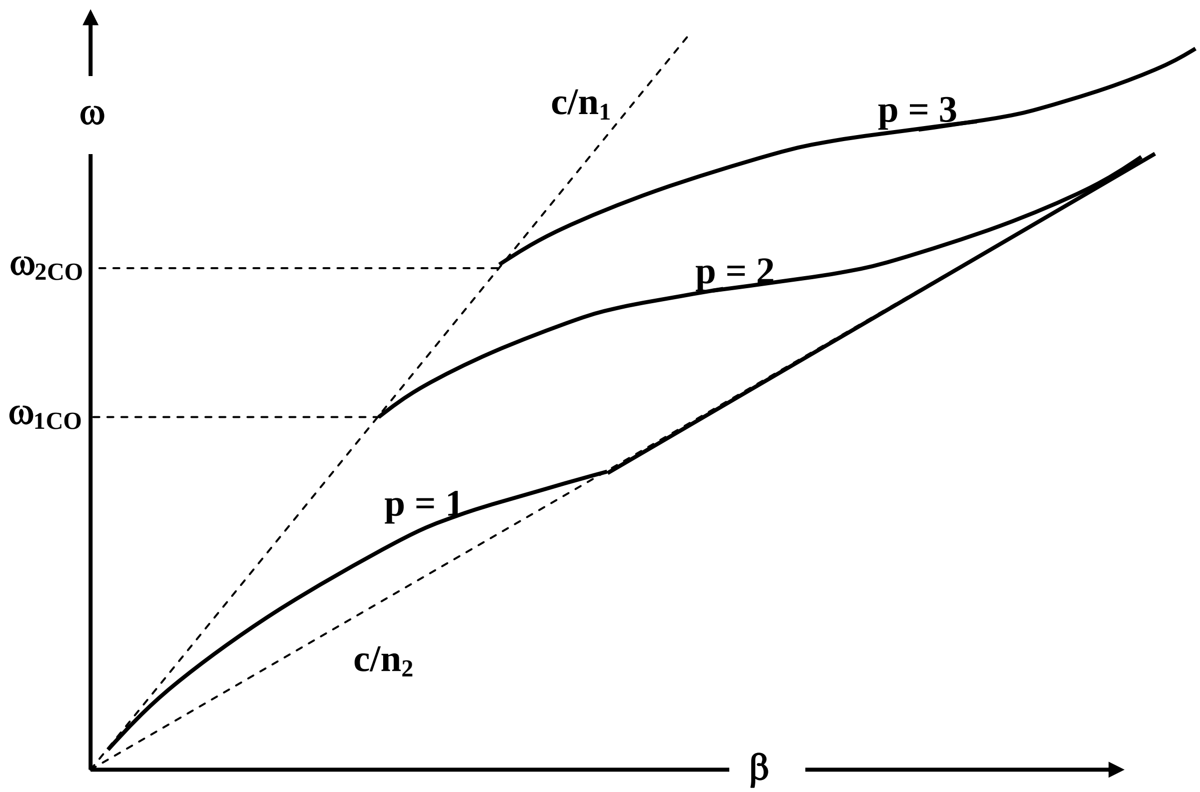
$$\omega = \frac{c}{n_G}\beta \tag{3.39b}$$

Again in this limit the phase and group velocity

$$v_P = \frac{c}{n_G} \qquad\qquad v_G = \frac{\partial \omega}{\partial k} = \frac{c}{n_G}$$

This is again as we should expect as at high frequencies $\alpha \to \infty$ and consequently most of the mode is confined to the guiding layer and will travel with a velocity $\frac{c}{n_G}$.

In between these two limits the ω, β curve will lie between the two lines of slope $\frac{c}{n_G}$ and $\frac{c}{n_C}$.



The dispersion relation for the first three modes showing the cutoff frequencies.
The dashed lines are straight lines with slope c/n_2 and c/n_1.

Numerical Aperture.

Up until now we have assumed the guide region to be the source region ignoring one small matter, how did the light get there? If light is to be guided within a waveguide (dielectric slab or fibre) it has to be incident at the interface between the high index guide and the low index cladding at an angle greater than the critical angle when it will undergo total internal reflection.

This immediately presents a problem as all rays are equally good if drawn running backwards (Fermats Principle). It is clear that there is no ray that will allow light from outside the guide to enter the guide AND be incident at an angle greater than the critical angle when it next arrives at the interface so how did a guided wave come to be in the high index region in the first place?

There are at least three ways around the problem of coupling light to a guide from an external source

(i) Light may be coupled from one guide or fibre to another using the evanescent fields that penetrate into the cladding.

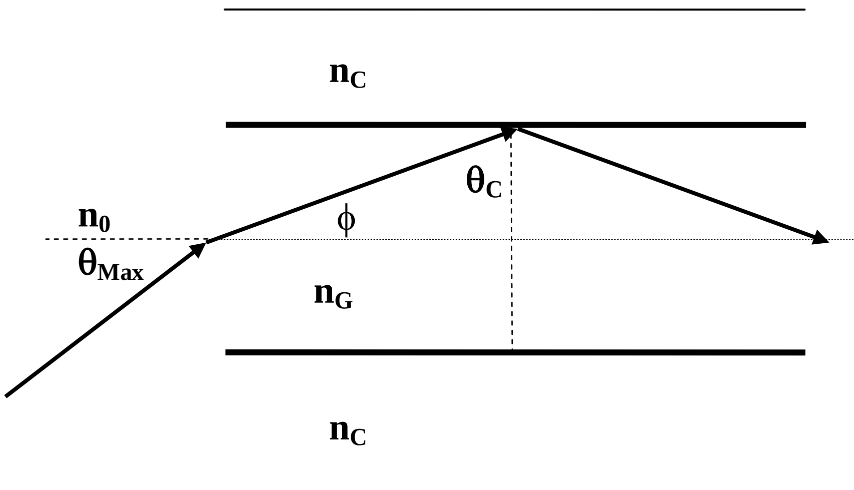
(ii) By using the evanescent fields associated with total internal reflection from prisms.

(iii) By use of a diffraction grating etched on the surface of the slab guide.

The first (and most obvious?) technique however, and one which is applicable to waveguide and fibre, is to focus light into the end of a guide/fibre. Indeed with laser diodes which tend to have a divergent output (broad angular distribution) compared with other lasers, a lens for focusing may not be necessary.

To choose an appropriate lens for direct focus coupling it is important to consider the collecting efficiency of the guide/fibre. A quantity that is used to express this is the numerical aperture (NA) of the fibre. Indeed other optical systems from monochromators to cameras use the concept and the NA is related to the inverse of the f-number of a camera. We are interested in the NA in the context of guides.

Consider the diagram below where an external ray of light in a medium of refractive index n_0 (usually air) is introduced into a guide such that it is subsequently guided at the critical angle. The angle θ_{Max} that the ray makes with the normal to the guide is the largest angle compatible with guiding in the guide. Were it any larger then the internal angle would be less than the critical angle. The numerical aperture is defined such that the inverse sin of the NA is this maximum angle, $\sin^{-1}(\mathrm{NA}) = \theta_{Max}$.



The NA can be found by application of Snells law at the two interfaces of the diagram:

$$n_0 \sin\theta_{Max} = n_G \sin\phi \qquad (3.40)$$

$$n_G \sin\theta_C = n_C \qquad (3.41)$$

Further

$$\sin\phi = \cos\theta_C = \sqrt{1-\sin^2\theta_C} = \sqrt{1-\frac{n_C^2}{n_G^2}} \qquad (3.42)$$

Then using 3.42 in 3.40

$$\sin\theta_{Max} = \frac{n_G}{n_0}\sqrt{1-\frac{n_C^2}{n_G^2}} = NA \qquad (3.43)$$

Simplifying

$$NA = \frac{\sqrt{n_G^2 - n_C^2}}{n_0} \tag{3.44}$$

Often the external medium is air and n_0 = 1, in this case the numerical aperture of the guide/fibre is given by ;

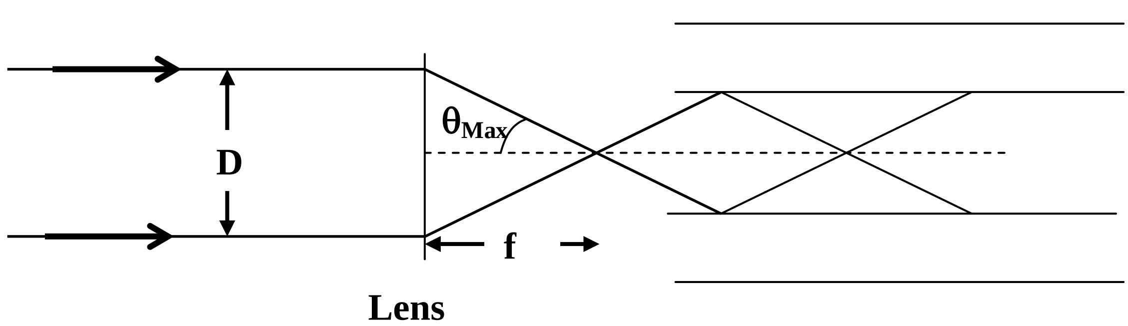
$$NA = \sqrt{n_G^2 - n_C^2} \tag{3.45}$$

It should be noted that the dimensions of the guide/fibre do not affect the numerical aperture.

Example.

For a highly collimated laser beam (parallel beam) with a beam diameter, *D*, of 2mm what choice of focal length lens, *F*, may be appropriate for a guide with numerical aperture 0.15?

The lens is required to produce a cone of light whose half angle, θ, is $\sin^{-1}0.15 = 8.6^0$ (see diagram below). At the focus of the laser beam a distance *F* from the lens (we have a parallel beam into the lens) the sin of the half angle would be approximately $\frac{D}{2F}$. To obtain optimum matching to the fibre a half angle of 8.6^0 would be appropriate. In other words $\frac{D}{2F} = NA = 0.15$ or $F = \frac{D}{2\times 0.15} = \frac{2mm}{0.3} = 6.66mm$.



The system of **laser beam plus lens** can be assigned a numerical aperture $\frac{D}{2F}$ and the numerical aperture of the lens system has been matched to that of the fibre.

A diode laser will emit a diverging beam with a divergence angle φ_{Div} and it could thus be assigned an NA = $\sin^{-1}\varphi_{Div}$. If this does not match the NA of the fibre that it is to be coupled to, then an intervening lens would be required in order to maximise the coupling.

In summary for slab waveguides;

The field internal to the guide is cosinusoidal with solutions separable into

$$E = exp(j\beta z)cos(k_x x) \qquad \text{Symmetric Modes}$$

$$E = exp(j\beta z)sin(k_x x) \qquad \text{Asymmetric Modes}$$

The field external to the guide (in the cladding) is exponentially decaying

$$E = E_{Int}\, exp(-\alpha x)$$

The guidance conditions for achieving a guided mode are

$$W = U\,tan\,U \qquad \text{Symmetric Modes}$$

$$W = -U\,cot\,U \qquad \text{Asymmetric Modes}$$

There is also a characteristic equation

$$W^2 + U^2 = V^2$$

Which used with the guidance conditions allows graphical solution of the guidance problem.

U and W are the Internal Normalised Transverse Wavevector and the External Normalised Transverse Wavevector respectively.

$$U = k_{Gx}\frac{d}{2} = \sqrt{k_G^2 - \beta^2}\,\frac{d}{2}$$

$$W = jk_{Cx}\frac{d}{2} = \alpha\frac{d}{2} = \sqrt{\beta^2 - k_C^2}\,\frac{d}{2}$$

$$V = \sqrt{n_G^2 - n_C^2}\,k_0\frac{d}{2}$$

The guide will become single mode when $V < \pi/2 = 1.57$

The guide can be characterised by a numerical aperture

$$NA = \sqrt{n_G^2 - n_C^2}$$