Queen Mary
University of London

B. Sc. Examination by course unit 2011

MTH5112 Linear Algebra I

Duration: 2 hours

Date and time: 26 May 2011, 2.30pm

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): Prof C-H. Chu

Question 1 **(a)** Given the following system of equations

$$\begin{array}{rcrcrcrcl} x_1 & + & 3x_2 & + & 2x_3 & - & x_4 & = & 1 \\ x_1 & + & 6x_2 & + & x_3 & + & 2x_4 & = & 1 \\ 3x_1 & + & 15x_2 & + & 4x_3 & + & 3x_4 & = & 3, \end{array}$$

write down the augmented matrix for the system and use Gaussian elimination to bring the augmented matrix to row echelon form. [9]

Further, find the solution set of the system. [6]

(b) Let A be an $n \times n$ real matrix. State a necessary and sufficient condition, in terms of the determinant of A, for A to be invertible. [2]

Find all cofactors C_{ij} $(i, j = 1, 2)$, and the inverse, of the real matrix

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

where $a \neq b$. [8]

Question 2 **(a)** Use Gauss-Jordan inversion to find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$ [9]

(b) Let V and W be real vector spaces and let $L : V \longrightarrow W$ be a mapping. Explain what is meant by saying that L is *linear*. [3]

Let $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the mapping given by

$$L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + z \\ y + z \\ x + y + z \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.$$

Show that L is linear. [5]

Find the matrix representing L with respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in the domain and codomain. [5]

Using **(a)**, or otherwise, find the inverse $L^{-1} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ of L. [3]

Question 3 **(a)** Let V be a subspace of the Euclidean space $\mathbb{R}^n$. Explain what is meant by saying that a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in V is

(i) *linearly independent*, [3]

(ii) a *spanning set* for V, [3]

(iii) a *basis* for V. [2]

(b) Let

$$A = \begin{pmatrix} 1 & -1 & 3 & 1 & 2 \\ 2 & -2 & 6 & 3 & 0 \\ 3 & -3 & 9 & 4 & 2 \end{pmatrix}.$$

Find a basis for the row space $\mathrm{row}(A)$ of A. [5]
Determine, giving an argument, the nullity of A. [3]

(c) Let V be a subspace of $\mathbb{R}^4$ spanned by the linearly independent vectors

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -2 \\ 1 \\ 4 \\ -3 \end{pmatrix}.$$

Using the Gram-Schmidt process, or otherwise, find an orthogonal basis for V. [9]

Question 4 Let A be a real $n \times n$ matrix and let λ be a real number.

(a) Explain what is meant by saying that λ is an *eigenvalue* of A. [2]

(b) Let $\mathbf{v} \in \mathbb{R}^n$. Explain what is meant by saying that $\mathbf{v}$ is an *eigenvector* of A corresponding to λ. Define the term *eigenspace* of A corresponding to λ (in $\mathbb{R}^n$). [5]

(c) Find all eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 6 & -10 \\ 0 & 3 & -5 \end{pmatrix}.$$ [6]

Further, find a basis for the eigenspace of A corresponding to *each* eigenvalue. [8]

(d) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. [4]

End of Paper