0.1 Revision from Geometry I

Recall that an $m \times n$ *matrix* A is a rectangular array of scalars (real numbers)

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} .$$

We write $A = (a_{ij})_{m\times n}$ or simply $A = (a_{ij})$ to denote an $m \times n$ matrix whose (i, j)-entry is a_{ij}, i.e. a_{ij} is the i-th row and in the j-th column.

If $A = (a_{ij})_{m\times n}$ we say that A has **size** $m \times n$. An $n \times n$ matrix is said to be **square**.

Example 0.1.1. If
$$A = \begin{pmatrix} 1 & 3 & 2 \\ -2 & 4 & 0 \end{pmatrix} ,$$
then A is a matrix of size 2×3. The $(1, 2)$-entry of A is 3 and the $(2, 3)$-entry of A is 0.

Definition 0.1.2 (Equality)**.** Two matrices A and B are **equal** and we write $A = B$ if they have the same size and $a_{ij} = b_{ij}$ where $A = (a_{ij})$ and $B = (b_{ij})$.

Definition 0.1.3 (Scalar multiplication)**.** If $A = (a_{ij})_{m\times n}$ and α is a scalar, then αA (the **scalar product of** α **and** A) is the $m \times n$ matrix whose (i, j)-entry is αa_{ij}.

Definition 0.1.4 (Addition)**.** If $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ then the **sum** $A + B$ of A and B is the $m \times n$ matrix whose (i, j)-entry is $a_{ij} + b_{ij}$.

Definition 0.1.5 (Zero matrix)**.** We write $O_{m\times n}$ or simply O (if the size is clear from the context) for the $m \times n$ matrix all of whose entries are zero, and call it a **zero matrix**.

Definition 0.1.6 (Matrix multplication)**.** If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times p$ matrix then the **product** AB of A and B is the $m \times p$ matrix $C = (c_{ij})$ with

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} .$$

Definition 0.1.7 (Identity matrix)**.** An **identity matrix** I is a square matrix with 1's on the diagonal and zeros elsewhere. If we want to emphasise its size we write I_n for the $n \times n$ identity matrix.

Matrix multiplication satisfies the following rules.

Theorem 0.1.8. *Assume that α is a scalar and that A, B, and C are matrices so that the indicated operations can be performed. Then:*

(a) $IA = A$ *and* $BI = B$*;*

(b) $A(BC) = (AB)C$*;*

(c) $A(B + C) = AB + AC$*;*

(d) $(B + C)A = BA + CA$*;*

(e) $\alpha(AB) = (\alpha A)B = A(\alpha B)$*.*

Notation

- Since $A(BC) = (AB)C$, we can omit the brackets and simply write ABC and similarly for products of more than three factors.
- If A is a square matrix we write $A^k = \underbrace{AA\cdots A}_{k \text{ factors}}$ for the k**-th power** of A.

$$\boxed{\textbf{Warning: } \text{In general } AB \neq BA!}$$

Definition 0.1.9. If A and B are two matrices with $AB = BA$, then A and B are said to **commute**.

Definition 0.1.10. If A is a square matrix, a matrix B is called an *inverse* of A if

$$AB = I \quad \text{and} \quad BA = I\,.$$

A matrix that has an inverse is called **invertible**

0.2 Transpose of a matrix

The first new concept we encounter is the following:

Definition 0.2.1. The *transpose* of an $m \times n$ matrix $A = (a_{ij})$ is the $n \times m$ matrix $B = (b_{ij})$ given by

$$b_{ij} = a_{ji}$$

The transpose of A is denoted by A^T.

Example 0.2.2.

$$\text{(a)} \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

$$\text{(b)} \quad B = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \Rightarrow B^T = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$$

Matrix transposition satisfies the following rules:

Theorem 0.2.3. *Assume that α is a scalar and that A, B, and C are matrices so that the indicated operations can be performed. Then:*

(a) $(A^T)^T = A$;

(b) $(\alpha A)^T = \alpha(A^T)$;

(c) $(A + B)^T = A^T + B^T$;

(d) $(AB)^T = B^T A^T$.

Theorem 0.2.4. *Let A be invertible. Then A^T is invertible and*

$$(A^T)^{-1} = (A^{-1})^T .$$

0.3 Special types of square matrices

In this section we briefly introduce a number of special classes of matrices which will be studied in more detail later in this course.

Definition 0.3.1. A matrix is said to be **symmetric** if $A^T = A$.

Note that a symmetric matrix is necessarily square.

Example 0.3.2.

$$\text{symmetric:} \quad \begin{pmatrix} 1 & 2 & 4 \\ 2 & -1 & 3 \\ 4 & 3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 5 & 2 \\ 2 & -1 \end{pmatrix}.$$

$$\text{not symmetric:} \quad \begin{pmatrix} 2 & 2 & 4 \\ 2 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Definition 0.3.3. A square matrix $A = (a_{ij})$ is said to be
upper triangular if $a_{ij} = 0$ for $i > j$;
lower triangular if $a_{ij} = 0$ for $i < j$;
diagonal if $a_{ij} = 0$ for $i \neq j$.

If $A = (a_{ij})$ is a square matrix of size $n \times n$, we call $a_{11}, a_{22}, \ldots, a_{nn}$ the **diagonal entries** of A.

0.4 Linear systems in matrix notation

Let

$$\mathbb{R}^n = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \middle| \; a_1, a_2, \cdots, a_n \in \mathbb{R} \right\}.$$

Suppose we are given an $m \times n$ linear system

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}. \tag{1}$$

We can reformulate this system into a single matrix equation.

$$A\mathbf{x} = \mathbf{b}, \tag{2}$$

where

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n, \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m,$$

and where $A\mathbf{x}$ is interpreted as the matrix product of A and $\mathbf{x}$.

Lemma 0.4.1. *Let A be an $m \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^m$. Suppose that M is an invertible $m \times m$ matrix. The following two systems have the same solution set:*

$$A\mathbf{x} = \mathbf{b} \tag{3}$$

$$MA\mathbf{x} = M\mathbf{b} \tag{4}$$

0.5 Elementary matrices and the Invertible Matrix Theorem

Definition 0.5.1. An **elementary matrix** of **type I** (respectively, **type II**, **type III**) is a matrix obtained by applying an elementary row operation of type I (respectively, type II, type III) to an identity matrix.

Example 0.5.2.

$$\text{type I: } E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{(take } I_3 \text{ and swap rows 1 and 2)}$$

$$\text{type II: } E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad \text{(take } I_3 \text{ and multiply row 3 by 4)}$$

$$\text{type III: } E_3 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{(take } I_3 \text{ and add 2 times row 3 to row 1)}$$

Let us now consider the effect of left-multiplying an arbitrary 3×3 matrix A in turn by each of the three elementary matrices given in the previous example.

Example 0.5.3. Let $A = (a_{ij})_{3\times 3}$ and let E_l $(l = 1, 2, 3)$ be defined as in the previous example. Then

$$E_1A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

$$E_2A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 4a_{31} & 4a_{32} & 4a_{33} \end{pmatrix},$$

$$E_3A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} + 2a_{31} & a_{12} + 2a_{32} & a_{13} + 2a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Theorem 0.5.4. *If E is an $m \times m$ elementary matrix obtained from I by an elementary row operation, then left-multiplying an $m \times n$ matrix A by E has the effect of performing that same row operation on A.*

Theorem 0.5.5. *If E is an elementary matrix, then E is invertible and E^{-1} is an elementary matrix of the same type.*

Proof. The assertion follows from the previous theorem and the observation that an elementary row operation can be reversed by an elementary row operation of the same type. More precisely,

- if two rows of a matrix are interchanged, then interchanging them again restores the original matrix;
- if a row is multiplied by $\alpha \neq 0$, then multiplying the same row by $1/\alpha$ restores the original matrix;
- if α times row q has been added to row r, then adding $-\alpha$ times row q to row r restores the original matrix.

Now, suppose that E was obtained from I by a certain row operation. Then, as we just observed, there is another row operation of the same type that changes E back to I. Thus there is an elementary matrix F of the same type as E such that $FE = I$. A moment's thought shows that $EF = I$ as well, since E and F correspond to reverse operations. All in all, we have now shown that E is invertible and its inverse $E^{-1} = F$ is an elementary matrix of the same type. $\square$

Example 0.5.6. Determine the inverses of the elementary matrices E_1, E_2, and E_3 in Example 0.5.2.

Solution. In order to transform E_1 into I we need to swap rows 1 and 2 of E_1. The elementary matrix that performs this feat is

$$E_1^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Similarly, in order to transform E_2 into I we need to multiply row 3 of E_2 by $\frac{1}{4}$. Thus

$$E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} .$$

Finally, in order to transform E_3 into I we need to add -2 times row 3 to row 1, and so

$$E_3^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

□

Definition 0.5.7. A matrix B is **row equivalent** to a matrix A if there exists a finite sequence $E_1, E_2, \ldots, E_k$ of elementary matrices such that

$$B = E_k E_{k-1} \cdots E_1 A .$$

In other words, B is row equivalent to A if and only if B can be obtained from A by a finite number of row operations.

The following properties of row equivalent matrices are easily established:

Remark 0.5.8. (a) A is row equivalent to itself;

(b) if A is row equivalent to B, then B is row equivalent to A;

(c) if A is row equivalent to B, and B is row equivalent to C, then A is row equivalent to C.

Theorem 0.5.9 (Invertible Matrix Theorem)**.** *Let A be a square $n \times n$ matrix. The following are equivalent:*

(a) *A is invertible;*

(b) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution;*

(c) *A is row equivalent to I;*

(d) *A is a product of elementary matrices.*

0.6 Gauss-Jordan inversion

The Invertible Matrix Theorem provides a simple method for inverting matrices. Recall that the theorem states (amongst other things) that if A is invertible, then A is row equivalent to I. Thus there is a sequence $E_1, \ldots E_k$ of elementary matrices such that

$$E_k E_{k-1} \cdots E_1 A = I \, .$$

Multiplying both sides of the above equation by A^{-1} from the right yields

$$E_k E_{k-1} \cdots E_1 = A^{-1} \, ,$$

that is,

$$E_k E_{k-1} \cdots E_1 I = A^{-1} \, .$$

Thus, the same sequence of elementary row operations that brings an invertible matrix to I, will bring I to A^{-1}. This gives a practical algorithm for inverting matrices, known as Gauss-Jordan inversion.

Gauss-Jordan inversion
Bring the augmented matrix $(A\|I)$ to reduced row echelon form. If A is row equivalent to I, then $(A\|I)$ is row equivalent to $(I\|A^{-1})$. Otherwise, A does not have an inverse.

Example 0.6.1. Show that

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{pmatrix}$$

is invertible and compute A^{-1}.

Solution. Using Gauss-Jordan inversion we find

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 0 & 3 & 8 & 0 & 0 & 1 \end{array}\right) \sim R_2 - 2R_1 \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 3 & 8 & 0 & 0 & 1 \end{array}\right)$$

$$\underset{R_3 - 3R_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & -1 & 6 & -3 & 1 \end{array}\right) \underset{(-1)R_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & -6 & 3 & -1 \end{array}\right)$$

$$\sim R_2 - 3R_3 \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 16 & -8 & 3 \\ 0 & 0 & 1 & -6 & 3 & -1 \end{array}\right) \overset{R_1 - 2R_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -31 & 16 & -6 \\ 0 & 1 & 0 & 16 & -8 & 3 \\ 0 & 0 & 1 & -6 & 3 & -1 \end{array}\right).$$

Thus A is invertible (because it is row equivalent to I_3) and

$$A^{-1} = \begin{pmatrix} -31 & 16 & -6 \\ 16 & -8 & 3 \\ -6 & 3 & -1 \end{pmatrix}.$$

$\square$