Section 2.

Curvilinear Motion

The study of the motion of a body along a *general* curve.

We define $\hat{u}_T$ the unit vector at the body, tangential to the curve

$\hat{u}_N$ the unit vector normal to the curve

Clearly, these unit vectors change with time, $\hat{u}_T(t), \hat{u}_N(t)$

But, their lengths are always $|\hat{u}_T(t)| = 1, |\hat{u}_N(t)| = 1$

And, we can always write a vector $\vec{u}$ as $\vec{u} = u_T\hat{u}_T + u_N\hat{u}_N$

The *velocity* $\vec{v}$ is always tangential to the curve,

$$\vec{v} = v(t)\hat{u}_T(t)$$

The *acceleration* $\vec{a}$ is not always tangential to the curve:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(v(t)\hat{u}_T(t)\right)$$

$$= \hat{u}_T\frac{dv}{dt} + v\frac{d\hat{u}_T}{dt}$$

But what is $\frac{d\hat{u}_T}{dt}$?

For straight-line motion, $\hat{u}_T$ is constant, i.e. $\frac{d\hat{u}_T}{dt} = 0$.

Otherwise, $\hat{u}_T$ changes *direction* (not magnitude, always 1).

Let the path of the body include a small arc length ds from the point A to the nearby point A', turning through a small angle dφ. That is, $\hat{u}'_T$ at A' makes the angle dφ with $\hat{u}_T$ at A.

The change in $\hat{u}_T$ *is* $\underline{d\hat{u}_T = \hat{u}'_T - \hat{u}_T = d\varphi\,\hat{u}_N}$

So, $\frac{d\hat{u}_T}{dt} = \frac{d\varphi}{dt}\hat{u}_N$

But $\underline{\frac{d\varphi}{dt} = \frac{d\varphi}{ds}\frac{ds}{dt} = v\frac{d\varphi}{ds}}$

The normals to the curve at A and A' meet at a point C. The distance to C defines the radius of curvature R of the arc.

Then $ds = Rd\varphi$, $\frac{d\varphi}{ds} = \frac{1}{R}$, $\frac{d\varphi}{dt} = \frac{v}{R}$

and $\frac{d\hat{u}_T}{dt} = \frac{v}{R}\hat{u}_N$

Hence
$$\boxed{\vec{a} = \frac{dv}{dt}\hat{u}_T + \frac{v^2}{R}\hat{u}_N}$$

That is, the acceleration has radial and tangential components:

$$a_R = \frac{v^2}{R}$$
$$a_T = \frac{dv}{dt}$$

The radial component changes the direction of the velocity.

- For uniform motion along the curve, $a_T = 0$, so body moves at constant *speed.* Velocity varies, with $a_R \neq 0$
- For rectilinear motion, $a_R = 0$, radius R is ∞.
- These results are needed for planetary motion – a particularly important application.

Circular Motion

Specialise to case where path is circular.

Since $\vec{v}$ is always tangential, $\vec{v} \perp$ radial direction

$$v = \frac{ds}{dt} = \frac{Rd\theta}{dt} = R\omega$$

ω is *angular velocity*, radians per second.

In vector notation:

Let $\vec{r}$ be the position vector of the body from an arbitrary point on the axis, so that angle between $\vec{r}$ and the axis $\vec{\omega}$ is γ. Then

$$r = |\vec{r}|, \quad \omega = |\vec{\omega}|$$
$$R = r\sin\gamma$$
$$v = |\vec{v}| = R\omega = \omega r\sin\gamma$$

and so $\boxed{\vec{v} = \vec{\omega}\times\vec{r}}$

(Note - $\vec{\omega}$ is defined to have length ω)

Acceleration:

Tangential: $a_T = \frac{dv}{dt} = \frac{d}{dt} R\omega = R\frac{d\omega}{dt}$, because R constant.

So $$\boxed{a_T = R\alpha \quad \text{where } \alpha \equiv \frac{d\omega}{dt}}$$

Radial: $a_R = \frac{v^2}{R}$, $\quad or\ \vec{a}_R = \frac{v^2}{R}\hat{u}_N$

Hence *centripetal* or *centrifugal* force, from $F=ma$, is

$$F = \frac{mv^2}{R} = mR\omega^2$$

The acceleration $\vec{a}_N$ is due to the *centripetal* force exerted on the body to keep it moving in a circle. The centripetal force acts *centrally*, i.e. is always directed to the centre, and it is responsible for changing the direction of the motion. It does not change the magnitude.

As Action equals Reaction (Newton's Law) it is perfectly correct and often convenient to consider the *centrifugal* force, which is the force the body exerts.

Uniform circular motion:

$$a_T = \frac{dv}{dt} = 0 \qquad \text{Tangential acceleration}$$

$$\alpha = \frac{d\omega}{dt} = 0 \qquad \text{Angular acceleration}$$

In vector form:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\vec{\omega}\times\vec{r}\right)$$
$$= \underbrace{\frac{d\vec{\omega}}{dt}}_{=0}\times\vec{r} + \vec{\omega}\times\frac{d\vec{r}}{dt}$$

(uniform circular motion)

So $\vec{a} = \vec{\omega}\times\vec{v} = \underline{\underline{\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)}}$ *Centripetal acceleration in vector form*

Note: $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq (\mathbf{A}\times\mathbf{B})\times\mathbf{C}$

Mass on String

Mass m attached to centre by string length r, rotating in circle therefore of radius r, at angular velocity ω. Neglect gravity. We may immediately write down the tension in the string:

$$T = m\vec{a}_N = mR\omega^2$$

This is the (inward) centripetal force exerted by the string on the mass, responsible for the (inward) acceleration of the mass. We may also identify the (outward) force (reaction) exerted by the mass on the string, the centrifugal force, responsible for the tension in the string.

Other aspects of this problem will be investigated later.

Rotation of the Earth

The Earth rotates on its axis, with constant ω for the uniform circular motion of all points. Note $\omega = \dfrac{2\pi}{24\times60\times60} = 72.72\ \mu\text{rad}\,\text{sec}^{-1}$. Consider a point A on surface at latitude λ. The tangential velocity at A is

$$v = r\omega = \omega R\cos\lambda = 463\cos\lambda\ m\,s^{-1} = 1036\cos\lambda\ \text{mph}$$

where R is the radius of the Earth (6370 km) and r is the distance of A from the axis of rotation.

The centripetal acceleration is

$$a_N = R\omega^2 = r\omega^2\cos\lambda = 0.034\cos\lambda\ m\,s^{-2}$$

At the equator ($\lambda = 0$), this is 0.3% of g.

BANKED RAILWAY TRACK

On curves (radius r), railway track is banked ("superelevated") to supply centripetal force for trains running at speed v. What is the required angle α of bank? Required force is

$$F = \frac{mv^2}{r}$$

This must be the horizontal component of the normal reaction of track on train, i.e.

$$F = F_N \sin\alpha = \frac{mv^2}{r}$$

But the weight of the train must equal the vertical component of the normal reaction of track on train, i.e.

$$mg = F_N \cos\alpha$$

So

$$F_N = \frac{mg}{\cos\alpha} = \frac{mv^2}{r\sin\alpha}$$

$$\Rightarrow \tan\alpha = \frac{v^2}{gr}$$

For typical values, $v = 100$ mph, $r = 1$ mile,

$$\alpha = \arctan\frac{v^2}{gr} = \frac{\left(44.7\ m\,s^{-1}\right)}{9.81\ m\,s^{-2} \times 1609\ m} = 7^\circ$$

For the standard gauge of 4 ft 8½ in, this means the outer rail is lifted ("superelevated") seven inches above the inner rail.

Exercise: What do the passengers feel in a train which is stationary on this curve? In a train which goes round the curve at 150 mph?

UNIFORMLY ROTATING FRAMES OF REFERENCE

Consider a stationary frame of reference S, coordinates (x, y, z, t) and origin O, and a frame S', rotating about the z-axis at a constant angular velocity ω and with origin $O' = O$, which therefore has a coordinate system $(x', y', z' = z, t' = t)$.

We want to derive relationships between the quantities such as position, velocity and acceleration measured in S' and measured in S.

Consider a body at a point A' at rest in S'. Clearly in S the body is in circular motion and has a velocity

$$\vec{v} = \vec{\omega} \times \vec{r}$$

If, however, the body, the point A', moves at $\vec{v}'$ with respect to frame S', then vector addition of velocities gives its velocity in S as

$$\boxed{\vec{v} = \vec{v}' + \omega \times \vec{r}}$$

And its acceleration? [*Viewed from S*]

As always, we need only differentiate the velocity with respect to time to get the acceleration. This can be done with vectors. However, it is a tricky example of vector calculus and will be presented in MT2 (Semester B).

The body has an acceleration in S' which we call $\vec{a}'$. In S we see additionally the centripetal acceleration $-\omega^2 \vec{r}$. We see also the Coriolis acceleration if there is a radial component in $\vec{v}'$.

CORIOLOIS FORCE

Consider an air current flowing from the North Pole to the Equator. It starts off with no East-West velocity. As it flows south, the Earth turns under it to the East. (The sun rises in the East.) If it underwent no eastward acceleration, by the time it reached the Equator it would constitute a 1000mph East wind. From the Earth as a frame of reference, it would appear that large westward forces had been exerted on it. This is the Coriolis force. Like centrifugal force, it is termed *fictitious*.

The Coriolis acceleration can be derived without vector calculus:

Let the body move outwards radially in S' from the centre at $t = 0$ to a point P' at a radius r at time t. Then its radial velocity is

$$v'_R = \frac{r}{t}$$

From the point of view of S it started moving towards a point P coincident with P' at $t = 0$. When it reaches P', that point is now a distance ωrt away from P tangentially. The body started with no tangential velocity (at the centre. So from the point of view of S it has accelerated tangentially, and using

$$s = ½ a_T t^2$$

and putting in the values, we have

$$a_T = \frac{2s}{t^2} = \frac{2\omega r t}{t^2} = \frac{2\omega r}{t} = \frac{2\omega r v_R}{r} = 2\omega v_R$$

The Coriolis force is tangential,and independent of radius, so it acts even at the centre.

MOTION RELATIVE TO EARTH

Falling Bodies – Centrifugal Term:

Let g_0 be acceleration due to gravity if Earth didn't rotate (i.e. gravity as viewed from S).

Then the effective gravity, gravity as viewed from the Earth's rotating frame S', is

$$\vec{g}_e = \vec{g}_0 - \vec{\omega} \times (\vec{\omega} \times r')$$

These are not parallel, with $\vec{g}_0$ pointing towards the centre of the Earth, and the centrifugal term pointing outwards from the Earth;' axis. So gravity is reduced and tilted towards the Equator.

Bodies falling towards the ground in the Northern hemisphere are displaced to the South, while bodies falling in the Southern hemisphere are displaced to the North.

The displacement vanishes at the Poles and at the Equator.

Coriolis Term:

That only applies to bodies that have *no velocity* in the Earth's frame. Let the body be falling vertically at velocity v'. Then Coriolis term is $-2\omega \times \vec{r}$, which points East in both hemispheres.

The displacement vanishes at the Poles and is maximum at the Equator.

Bodies with Tangential Velocity:

Northward in Northern hemisphere, $-2\vec{\omega} \times v'$ points to the East, and the motion is deflected to the East. Northwards in Southern hemisphere, $-2\omega' \times \vec{v}'$ points to the West and the motion is deflected to the West.

The effect is maximum at the Poles and vanishes at the Equator.

Consequences of the Coriolis Froce for the Weather:

1. **Cyclones.** A region at low pressure tends to fill as air flows in radially, at right angles to the contours of constant pressure (isobars). The Coriolis force deflects the radial motion, to the right in Northern hemisphere (and to the left in Southern hemisphere). This sets up an anti-clockwise rotation (clockwise in Southern). A cyclone becomes stable when the air flow is parallel to the isobars. *Look for this on weather maps.*

2. **Trade Winds.** The largest scale pattern in the atmosphere is the convection of heat from the Equator to the Poles, with cold air returning South at sea-level. This current of air is deflected to the West (in both hemispheres), so that the most stable wind patterns are the Trade Winds, a North-East wind in the Northern hemisphere and a South-East wind in the Southern hemisphere. *Look for these on weather maps.*