## Relativistic Kinetics

## Velocity:

Let a body be travelling at velocity $u^{\prime}$ in the $x^{\prime}$-direction of frame $S^{\prime}$, having started at the origin at $t^{\prime}=0$. What is its velocity in $S$ ?

Consider that $u^{\prime}=\frac{x^{\prime}}{t^{\prime}}$. The Lorentz Transform gives,

$$
\begin{aligned}
& x=\gamma x^{\prime}+v \gamma t^{\prime} \\
& t=\frac{v}{c^{2}} \gamma x^{\prime}+\gamma t^{\prime} \\
& u=\frac{x}{t}=\frac{\gamma x^{\prime}+v \gamma t^{\prime}}{\frac{v}{c^{2}} \gamma x^{\prime}+\gamma t^{\prime}}=\frac{\frac{x^{\prime}}{t^{\prime}}+v}{\frac{v}{c^{2}} \frac{x^{\prime}}{t^{\prime}}+1}=\frac{u^{\prime}+v}{1+\frac{u^{\prime} v}{c^{2}}}
\end{aligned}
$$

Addition of Velocities: This gives us a formula for the addition of velocities. Let a spaceships be travelling at $v$ relative to Earth. Treat the spaceship's frame as $S^{\prime}$ and let it send a probe forward at a speed relative to the spaceship of $u^{\prime}$. Then the speed of the probe relative to Earth $(S)$ is just that given by the formula.

Similarly, let a pair of twins travel in spaceships in opposite directions, both at speed $v$ relative to Earth $(v<c)$. Then we may take one twin as $S$, Earth as $S^{\prime}$ at $v$ relative to the first twin, and the second twin at $v$ relative to Earth. Then the velocity of the twins relative to each other is

$$
u=\frac{2 v}{1+\frac{v^{2}}{c^{2}}}
$$

Note that the combined velocity is always less than $c$. We may write,

$$
\begin{aligned}
& \frac{2 v}{1+\frac{v^{2}}{c^{2}}}<c \\
& \frac{2 v c^{2}}{c^{2}+v^{2}}<c \\
& 2 v c^{2}<c^{3}+v^{2} c \\
& 2 v c<c^{2}+v^{2} \\
& 0<c^{2}-v c+v^{2}-v c \\
& 0<c(c-v)-v(c-v) \\
& 0<c-v \\
& v<c
\end{aligned}
$$

which is true. This is one of the reasons that nothing can travel faster than light. No addition of velocities $<c$ can reach a total equal to $c$.

## Transforms of Velocity and Acceleration.

We may find what the velocity $u^{\prime}$ relative to the origin of $S^{\prime}$ is in $S$. We have $u^{\prime}=\frac{d x^{\prime}}{d t^{\prime}}$. The proper length $d x^{\prime}$ transforms to $\frac{d x}{\gamma}$ and the proper time $d t^{\prime}$ transforms to $\gamma d t$. So $u^{\prime}$ transforms to $u=\frac{u^{\prime}}{\gamma^{2}}$.

Similarly $a^{\prime}=\frac{d u^{\prime}}{d t^{\prime}}$, bringing another factor of $\gamma$. So $a=\frac{a^{\prime}}{\gamma^{3}}$.

## Mass and Momentum

A rigorous derivation of the transforms of mass and momentum between frames is very tedious. However, it should not be surprising that if a constant force adds incrementally less and less speed, it must be that the mass of the body is increasing. That is so*, and the dependence of mass on velocity relative to $S$ is just

$$
m(v)=\gamma m_{0}
$$

for a body stationary in $S^{\prime}$ with rest mass $m_{0}$ in $S^{\prime}$

Then the relativistic momentum vector is

$$
\mathbf{p}=\gamma m \mathbf{v}
$$

* You may look this up in any textbook entitled Introduction to Relativity or similar (e.g. W.G.V. Rosser, Introductory Relativity (Butterworths, London, 1967). The method is to consider an elastic collision between two bodies with masses $m$, and view it from two frames (such as centre of mass frame and another frame $S^{\prime}$ moving at $v$ relative to it). Conservation of momentum is required, and masses are allowed to depend on velocity relative to the frame. One gets an expression for the ratio of the masses at two different speeds $u_{1}$ and $u_{2}$ in $S$

$$
\frac{m\left(u_{1}\right)}{m\left(u_{2}\right)}=\frac{\sqrt{u_{2}^{2}+\frac{\gamma^{2} v^{2}}{\left(1-\gamma^{2}\right)}}}{\sqrt{u_{1}^{2}+\frac{\gamma^{2} v^{2}}{\left(1-\gamma^{2}\right)}}}
$$

However, it is not possible for the ratio of the masses in the unprimed frame to depend on our choice of primed frame and value of $v$. Consequently, this expression must be independent of $v$. This condition is sufficient to require that $\gamma$ has the same value as usual, and then the equation simplifies to

$$
\frac{m\left(u_{1}\right)}{m\left(u_{2}\right)}=\frac{\sqrt{u_{2}^{2}-c^{2}}}{\sqrt{u_{1}^{2}-c^{2}}}=\frac{\sqrt{1-\frac{u_{2}^{2}}{c^{2}}}}{\sqrt{1-\frac{u_{1}^{2}}{c^{2}}}}
$$

For this to hold for all values of $u$, it must be that

$$
m(u)=\frac{m_{0}}{\sqrt{1-\frac{u^{2}}{c^{2}}}}
$$

Force is always as given by Newton's Law, $F=\frac{d}{d t} p=\frac{d}{d t} \frac{m_{0} v}{\sqrt{1-\frac{v^{2}}{c^{2}}}}=\frac{m_{0} a}{\left(1-\frac{v^{2}}{c^{2}}\right)^{3 / 2}}$

$$
\text { Or, } \quad a=\frac{F}{m_{0}}\left(1-\frac{v^{2}}{c^{2}}\right)^{3 / 2}
$$

Again, we see that under a constant force, a body accelerates asymptotically towards the speed of light, rather than to infinite speed.

## Kinetic Energy

This is obtaining by integrating the work needed to accelerate a body from stationary to speed $v$. Let it be at speed $u$, then the work done to accelerate it to $u+d u$ is

$$
d W=F d x=\frac{m_{0} a}{\left(1-\frac{u^{2}}{c^{2}}\right)^{3 / 2}} d x
$$

We change variable using

$$
a d x=\frac{d u}{d t} d x=d u \frac{d x}{d t}=u d u
$$

Then the kinetic energy at the speed $v$ is the integral of the work from stationary to $v$ :

$$
\begin{aligned}
& K=\int_{0}^{v} d W=\int_{0}^{v} \frac{m_{0}}{\left(1-\frac{u^{2}}{c^{2}}\right)^{3 / 2}} u d u=\frac{m_{0} c^{2}}{\sqrt{1-\frac{v^{2}}{c^{2}}}}-m_{0} c^{2} \\
&=(\gamma-1) m_{0} c^{2} \\
& \begin{array}{l}
E=K+m_{0} c^{2} \\
E=m c^{2}
\end{array}
\end{aligned}
$$

A series expansion of these expressions yields $1 / 2 m v^{2}$ as the first term in the kinetic energy.

## Some Numbers

Hydrogen: Atomic weight 1, rest mass $\frac{1 g m}{\text { Avogadro's number }}$ $=1.66 \times 10^{-27} \mathrm{~kg}$

Energy $E=m_{0} c^{2}=1.66 \times 10^{-27} \times\left(3 \times 10^{8}\right)^{2}=1.5 \times 10^{-10} \mathrm{~J}$
At the atomic scale, energy is better measured in $\mathrm{eV}=\frac{1 J}{N_{C}}$ where $N_{C}$ is the number of electrons in a Coulomb, $N_{C}=\frac{1}{1.602 \times 10^{-19}}$. Then the mass or energy of the hydrogen atom is

## $938 \mathrm{MeV} \sim 1 \mathrm{GeV}$

At the Large Hadron Collider, Geneva (CERN), protons will be accelerated to many GeV , so to many times their rest mass.

Helium: Using the carbon-12 scale, the atomic weight of He is 4.0026 . On the same scale, the atomic weight of hydrogen is 1.00797 . So if four hydrogen atoms are combined (by nuclear fusion) to make one helium atom, there is a loss of mass of $4 \times 1.00797-4.0026=0.029$ atomic mass units. We saw above that 1 a.m.u. is about 1 GeV , so this loss is about 29 MeV .

Chemistry uses reactions (e.g. oxidation of carbon) which release a few eV per atom. So nuclear reactions are seven orders of magnitude more energetic.

