2. DIELECTRIC INTERFACE & FRESNEL'S EQUATIONS.

Dielectric Interface

The aim of these notes is to establish the following:

(i) Laws of Reflection and Refraction

(ii) Total Internal Reflecion, TIR, and the Critical Angle.

(iii) Evanescent Fields

(iv) Transverse Electric, TE, and Transverse Magnetic, TM, Polarisation.

(v) Reflection and Transmission Coefficients.

(vi) Brewster Angle.

Maxwell's Equations & Boundary Conditions

All of the physics of transmission and reflection of light at the interface between two dielectrics with differing refractive indices is determined by the boundary conditions on the electric and magnetic fields at that interface as required by Maxwells equations. We consider Maxwell's equations as they appear when written in integral form.

$$\oint \vec{E}.d\vec{A} = \frac{Q}{\varepsilon_0} \qquad (2.1)$$

$$\oint \vec{B}.d\vec{A} = 0 \qquad (2.1a)$$

$$\oint \vec{E}.d\vec{l} = -\frac{d\phi_B}{dt} \qquad (2.2)$$

$$\oint \vec{B}.d\vec{l} = \mu_0\left(i_C + \frac{d\phi_E}{dt}\right) \qquad (2.2a)$$

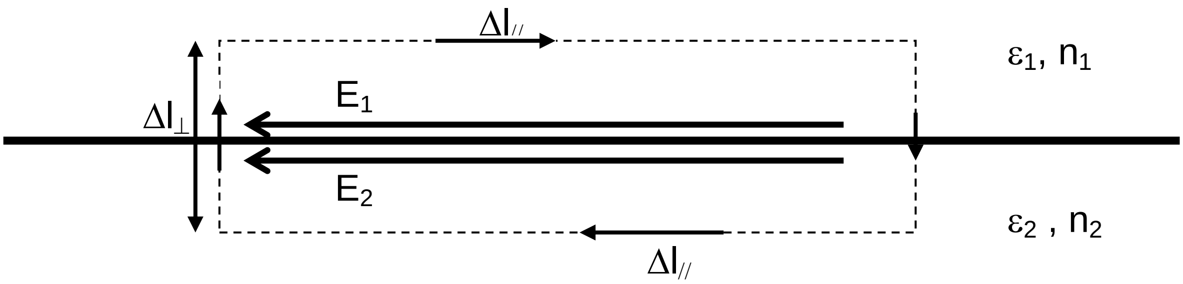
In particular we focus on just two of these

$$\oint_C E.dl = -\frac{\partial}{\partial t}\oint_S B.dA \qquad (2.2)$$

and

$$\oint_C H.dl = \frac{\partial}{\partial t}\oint_S D.dA$$

These are the integral forms of the Maxwell equations involving $\nabla \times E$ and $\nabla \times H$ using Stokes theorem and assuming no currents, $\boldsymbol{J} = 0$.

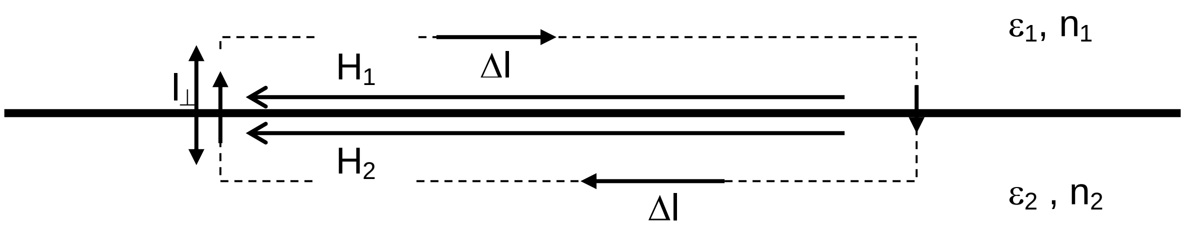


Consider the boundary between the two dielectrics shown above with the dashed contour crossing the boundary. As dA can be shrunk to zero by reduction of $l_\perp$.
the integral around the contour can be reduced to

$$\oint_C E \bullet dl = E_1 \bullet \Delta l_{//} + E_2 \bullet \Delta l_{//} = \left(-E_{1Tan} + E_{2Tan}\right)\Delta l_{//} = 0 \qquad (2.3)$$

$$E_{1_{Tan}} = E_{2_{Tan}} \qquad (2,3a)$$

And the same can be done with the H field



$$\oint_C H \bullet dl = H_1 \bullet \Delta l_{//} + H_2 \bullet \Delta l_{//} = \left(-H_{1Tan} + H_{2Tan}\right)\Delta l_{//} = 0 \qquad (2.4)$$

$$H_{1_{Tan}} = H_{2_{Tan}}$$

In other words the boundary condition we are looking for is:

The tangential <u>H</u> and <u>E</u> fields must be continuous across a boundary.

The other pair of Maxwell equations in integral form can be used to show that the perpendicular components of ***<u>D</u>*** and ***<u>B</u>*** at the interface must be continuous but these will concern us no further.

Laws of Reflection and Refraction

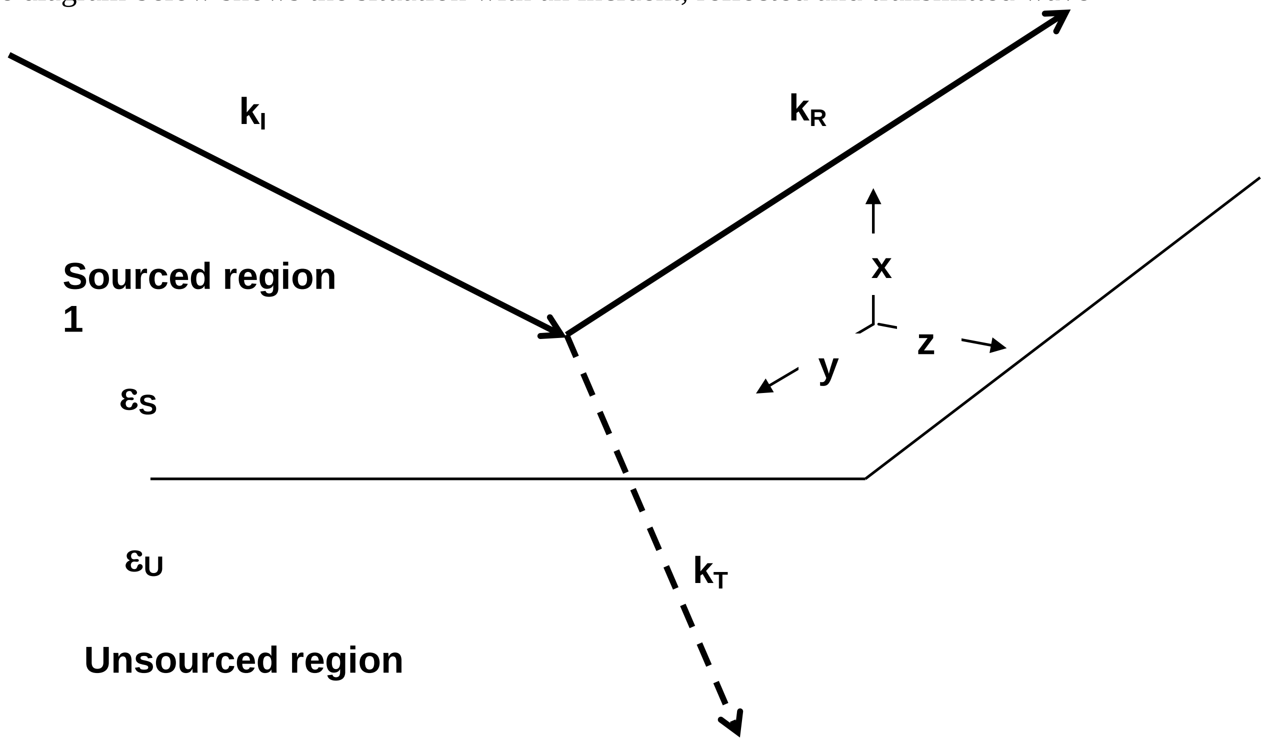
The first thing we can establish with these boundary conditions are the laws of reflection and refraction. To deal with this we need to understand the implications of the boundary conditions

Let the dielectric interface be in the *yz* plane through *x* = 0 as shown (below). The boundary conditions, as previously, require that all tangential fields (***E*** and ***H***) be continuous across the boundary at *x* = 0. Keeping vector notation for the time being we can write the ***E*** fields as:

$$\left\{\begin{matrix} E_I = A_I\, exp(-jk_I \bullet r) \\ E_R = A_R\, exp(-jk_R \bullet r) \\ E_T = A_T\, exp(-jk_T \bullet r) \end{matrix}\right\} \tag{2.5}$$

A's are the amplitudes of the incident, reflected and transmitted electric fields and ***k***'s are the wavevectors of those waves, denoted *I, R* and *T* respectively.

The diagram below shows the situation with an incident, reflected and transmitted wave



Please note in the above diagram a sourced region (where the light source is) and an unsourced region are indicated. In these notes the properties of the sourced region such as refractive index are generally denoted by with the subscript S, eg. ε_S , n_S *and the unsourced region by the subscript U, eg.* ε_U , n_U *etc*

The boundary condition then requires:

$$\left\lfloor A_I \, exp\left(-jk_{Iy}y\right)exp\left(-jk_{Iz}z\right)+A_R \, exp\left(-jk_{Ry}y\right)exp\left(-jk_{Rz}z\right)\right\rfloor_{Tan}=\left\lfloor A_T \, exp\left(-jk_{Ty}y\right)exp\left(-jk_{Tz}z\right)\right\rfloor_{Tan} \tag{2.6}$$

The important thing to note is that this must be true at all values of y and z and that matching at one point is insufficient.

This strict requirement that matching occurs everywhere on the interface then translates as:

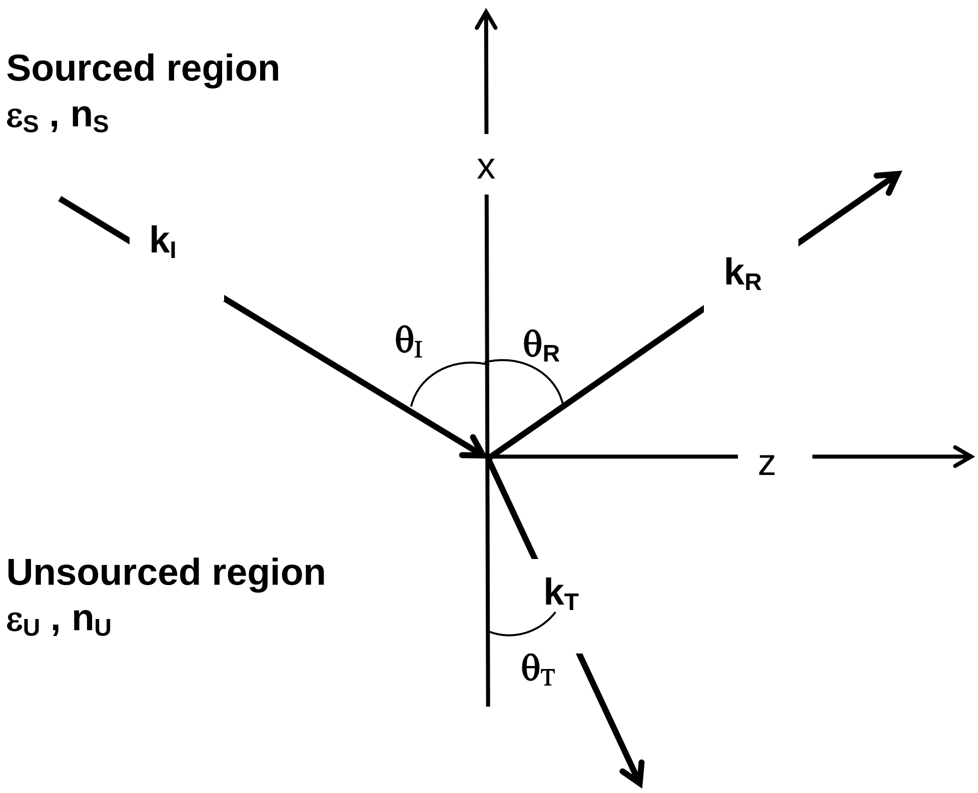
$$\left.\begin{array}{l} k_{Iy}=k_{Ry}=k_{Ty}=k_y \\ \\ \text{and that} \\ \\ k_{Iz}=k_{Rz}=k_{Tz}=k_z \end{array}\right\} \tag{2.7}$$

i.e. $\underline{\boldsymbol{k}_I}$, $\underline{\boldsymbol{k}_R}$ and $\underline{\boldsymbol{k}_T}$ **<u>must all lie in the same plane</u>** known as

the plane of incidence.

if this were not the case the fields whilst satisfying the boundary conditions at x, y, z = 0 (say) would not satisfy the boundary condition at all values of y and z ie. everywhere on the yz plane at x = 0!!

It is now possible, without any loss of generality, to rotate the co-ordinate system about the *x* axis to cause the *xz* plane to coincide with the plane of incidence. This will allow a simpler analysis (and simpler diagrams!).



With the axis now rotated we can establish some facts concerning reflection and transmission at oblique incidence using the boundary conditions.

Looking at the above diagram we only have *x* and *z* components of *k* for incident, reflected and transmitted waves and we have:

$$\left.\begin{aligned} k_I &= -\hat{x}k_{Ix} + \hat{z}k_{Iz} \\ k_R &= \hat{x}k_{Rx} + \hat{z}k_{Rz} \\ k_T &= -\hat{x}k_{Tx} + \hat{z}k_{Tz} \end{aligned}\right\} \qquad (2.8)$$

Note the sign of the unit vectors $\hat{x}$ and $\hat{z}$ which tells us about the direction of the wavevector.

Also from examining the diagram;

$$\left.\begin{aligned} k_{Ix} &= k_S \cos\theta_I & k_{Iz} &= k_S \sin\theta_I \\ k_{Rx} &= k_S \cos\theta_R & k_{Rz} &= k_S \sin\theta_R \\ k_{Tx} &= k_U \cos\theta_T & k_{Tz} &= k_U \sin\theta_T \end{aligned}\right\} \qquad (2.9)$$

k_S and k_U are the magnitudes of the wavevectors in the sourced and unsourced regions respectively

$$k_S = \frac{\omega}{v_S} = n_S \frac{\omega}{c} = n_S k_0 \qquad \text{and} \qquad k_U = \frac{\omega}{v_U} = n_U \frac{\omega}{c} = n_U k_0 \qquad (2.10)$$

Comparing the incident and transmitted waves the boundary conditions that the k_z vectors are equal give

$$k_S \sin\theta_I = k_U \sin\theta_T \qquad \text{or} \qquad n_S k_0 \sin\theta_I = n_U k_0 \sin\theta_T$$

In other words $\quad n_S \sin\theta_I = n_U \sin\theta_T \quad$ **<u>Snell's Law</u>** (2.11)

Comparing the incident and reflected waves the conditions give

$$k_{Iz} = k_{Rz} \qquad \text{or} \qquad n_S k_0 \cos\theta_I = n_S k_0 \cos\theta_R$$

In other words

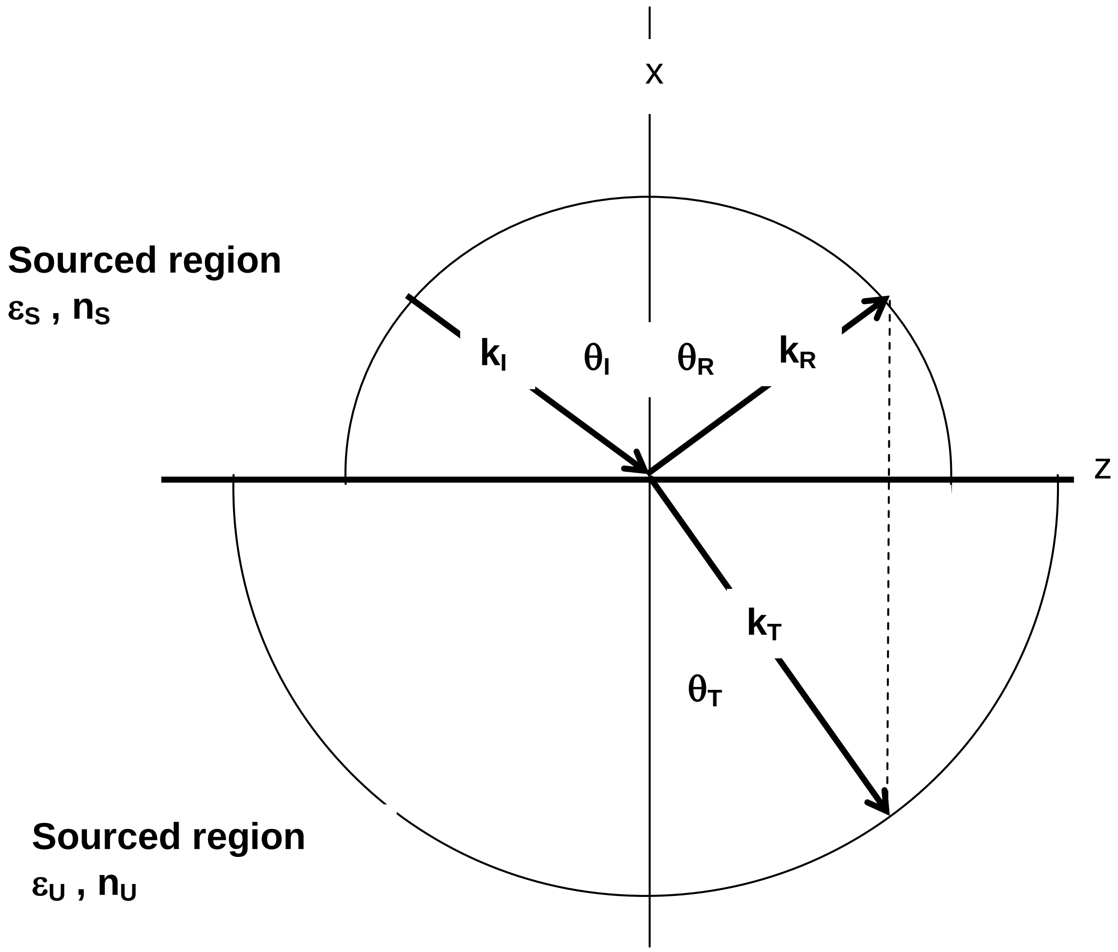
$$\cos\theta_I = \cos\theta_R \quad \Rightarrow \quad \theta_I = \theta_R$$ **<u>Law of reflection</u>** (2.12)

In everything that has been done in this section there has been one guiding principle and that is the boundary condition established early on through Maxwell's equations that the tangential fields match at all points along the boundary. We continue using that principle to obtain several other useful pieces of information.

Internal and external reflection.

We pause here to note that it is useful to distinguish between the situation where the sourced region has a greater refractive index than the unsourced region and vice versa. Snells law, 2.11, tells us that when the refractive index of the sourced region is greater than the unsourced region, the refracted (transmitted) ray is bent away from the normal whilst the opposite is true when the refractive index of the sourced region is less than that of the unsourced region. Bearing this in mind we can look at the two immediately preceding diagrams and realise that $n_S < n_U$. As far as reflection is concerned it is usual to refer to reflection occuring within a high refractive index region at the boundary with a lower refractive index region as ***internal reflection*** whereas when the reflection occurs within the lower refractive index region this is called ***external reflection***.

Total Internal Reflection, TIR

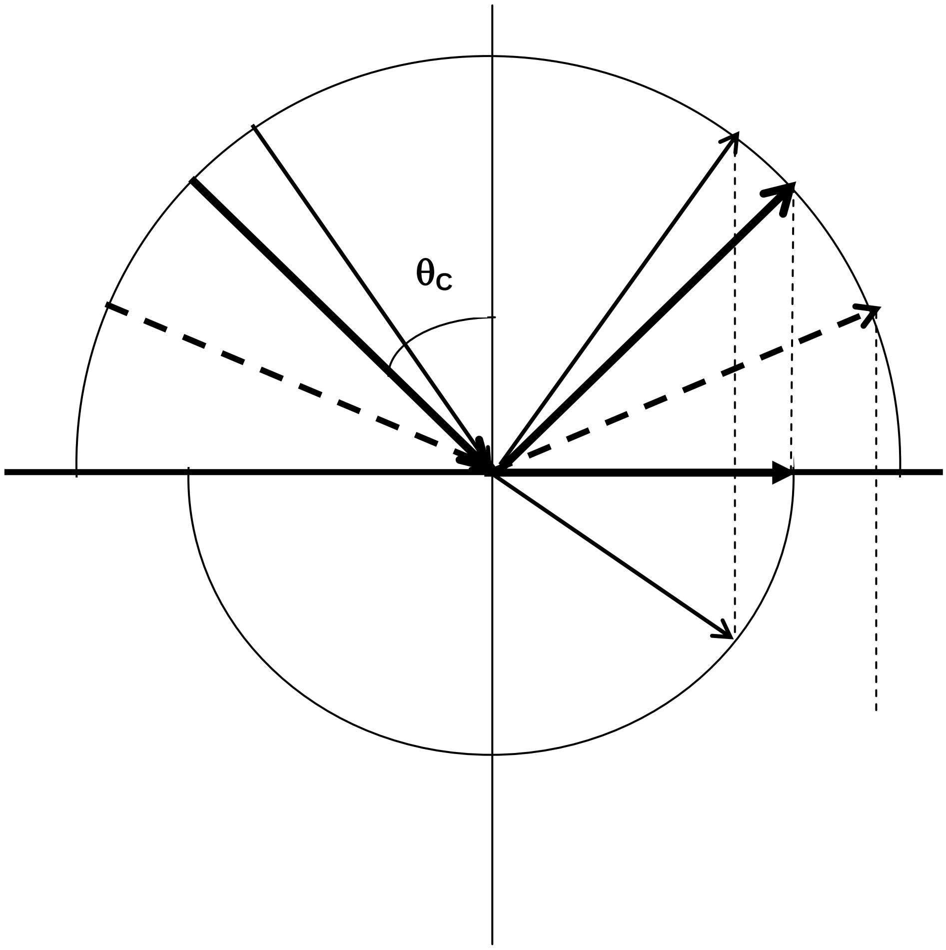


The construction shown above is known as a phase matching diagram. There is a boundary between two dielectrics along the z axis. The two concentric semicircles have radii equal to the magnitudes of the wavevectors, $n_S k_0$ and $n_U k_0$ in their respective regions.

Beginning with an incident wave represented by the arrow $\boldsymbol{k}_I$ travelling at some angle θ_I , we can use the law of reflection ($\theta_I = \theta_R$) to construct the arrow representing the reflected wave, $\boldsymbol{k}_R$. Note the arrows must have a length equal to the magnitude of the wavevector in the sourced region i.e. the radius of the semicircle in the sourced region and thus end on the semicircle.

We now construct the transmitted wave by dropping a perpendicular from the tip of the reflected arrow onto the circumference of the semicircle in region 2. This ensures that the ***z components*** of the wavevectors are equal in both regions when the transmitted wavevector is drawn to finish at the point of intersection of the perpendicular with the semicircle in the unsourced region.

In the example shown it is clear that the refractive index is greater in the unsourced region as the radius of the semicircle is greater there. More interesting is the situation in the phase diagram shown below where the refractive index in the sourced region is the greater.



We can now perform the same constructions for the incident wavevector and the reflected wavevector as shown by the three solid arrows. However if we make the angle of incidence a little larger (dashed line) we may no longer keep the z component of the wavevector in the unsourced region equal to that in the sourced region at least not yet! i.e. there is a critical angle, θ_C beyond which we can no longer satisfy the boundary condition ***with a wave transmitted into the unsourced region***. At this angle (thick arrows) there is a refracted wave travelling along the boundary and at greater angles (dashed arrows) there is only a reflected wave and no energy is transmitted into region 2. This phenomena is known as **Total Internal Reflection** and is the key to waveguiding and optical fibres.

To find the **critical angle** consider Snell's law,

$$n_S \sin\theta_I = n_U \sin\theta_T \tag{2.13}$$

or

$$\frac{n_S}{n_U}\sin\theta_I = \sin\theta_T \tag{2.14}$$

When $n_S > n_U$ and $\frac{n_S}{n_U} > 1$, because $\sin\theta_T$ (the RHS of the equation) can be no larger than 1, it is possible to find an angle where the LHS of the equation is greater than 1 and the equation cannot be (seemingly) satisfied. The critical angle, θ_C occurs where θ_T = 90^0 and $\sin\theta_T$ = 1. i.e.

$$\theta_C = \sin^{-1}\frac{n_U}{n_S} \tag{2.15}$$

Evanescent fields

There is in fact a way that the boundary conditions can be achieved for angles of incidence greater than the critical angle. ***<u>This does not lead to a field and thus energy propagating away from the interface</u>*** into region 2 but gives fields that are exponentially decaying away from the interface in region 2. These are of some technological interest and we examine them here.

The boundary condition is that the ***z* components** of the wavevectors match at the interface between sourced and unsourced regions. In other words given that;

$$k_{Sz}^2 + k_{Sx}^2 = k_S^2 \qquad \text{and} \qquad k_{Uz}^2 + k_{Ux}^2 = k_U^2$$

The boundary condition may be written as

$$k_{Sz} = \sqrt{k_S^2 - k_{Sx}^2} = \sqrt{k_U^2 - k_{Ux}^2} = k_{Uz} = k_z \tag{2.16}$$

This condition on the *z* components of *k* may be seen alternatively as a condition on the x component of the wavevector in the unsourced region by rewriting 2.16;

$$k_{Ux} = \sqrt{k_U^2 - k_z^2} = \sqrt{k_U^2 - k_S^2 \sin^2\theta_I} \tag{2.17}$$

Recall that the problem of satisfying the boundary condition arises because above the critical angle the z component in the sourced region is greater than the entire wavevector in the unsourced region i.e.

$$k_S \sin\theta_I > k_U$$

Inspection of 2.17 then shows that k_{Ux} is given by the square root of a negative number. In other words we have managed to satisfy the boundary condition by allowing k_{Ux} to be imaginary i.e.

$$k_{Ux} = \pm j\sqrt{k_S^2 \sin^2\theta_I - k_U^2} = \pm j\alpha \tag{2.18}$$

Where we have written

$$\alpha = \sqrt{k_S^2 \sin^2\theta_I - k_U^2} = k_0\sqrt{n_S^2 \sin^2\theta_I - n_U^2} \tag{2.19}$$

This gives us a spatially varying electric field when inserted back into our usual field description, exp(-jk.r) ,as follows

$$E(x,z) = A\exp\left((-jk_z z) + (\mp\sqrt{k_S^2 \sin^2\theta_I - k_U^2}\,)x\right) = A\exp(-jk_z z)\exp(\mp\alpha x) \tag{2.20}$$

In our case (see diagram and axes) the unsourced region goes into negative *x* co-ordinates and so we need to choose the positive solution in order to avoid an unphysical solution with an exponentially increasing field away from the boundary into the unsourced region.

$$E(x,z) = A\exp(\alpha x)\exp(-jk_z z) \tag{2.21}$$

This is a field with an exponentially decaying amplitude in the x direction away from the interface that is propagating in the z direction as a wave. The amplitude has dropped to $\frac{1}{e}$ of its value at the boundary after travelling a distance $x = \frac{1}{\alpha}$ away from the boundary into the unsourced region and the field is called an **evanescent field.** ***and exists only;***

i) ***when there is total internal reflection***

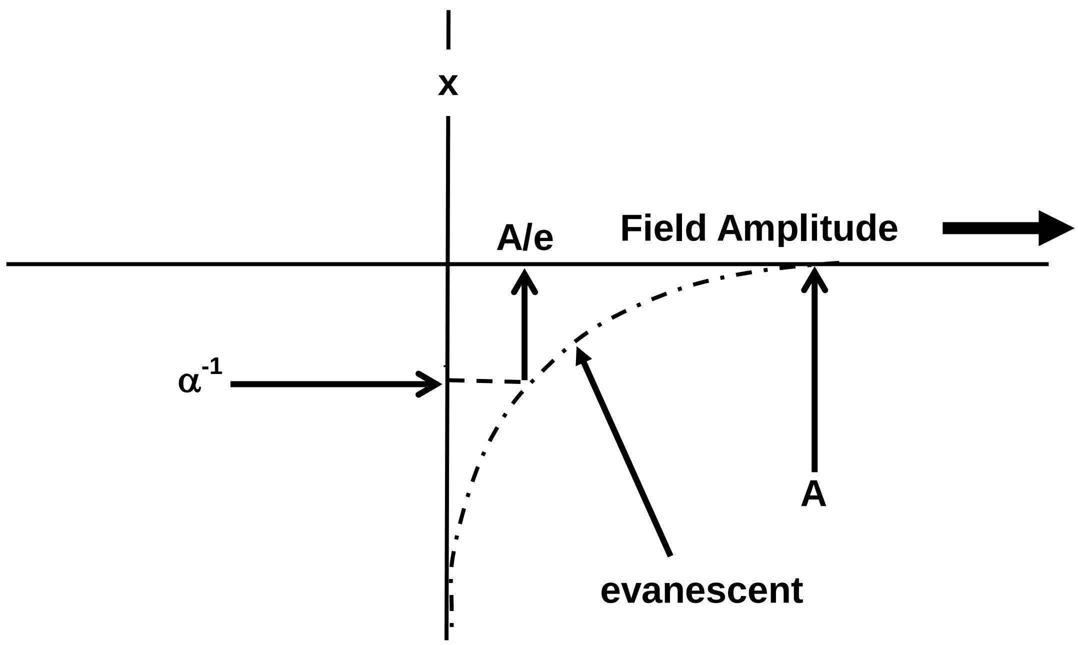
which only happens

ii) ***when the sourced region has a higher refractive index than the unsourced region***

and

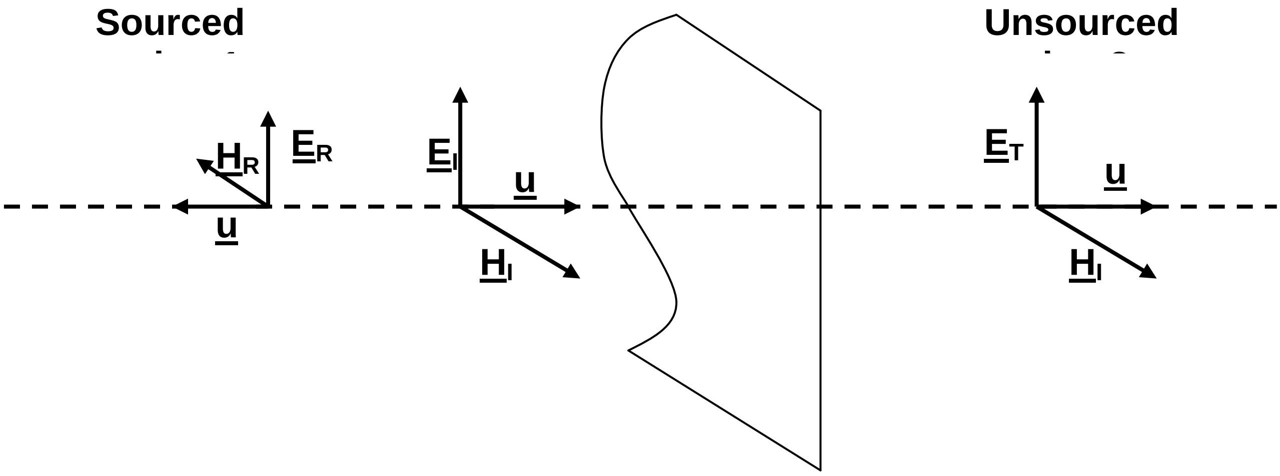
iii) ***when the angle of incidence is greater than the critical angle.***

The situation is analogous to an electron wave hitting a potential step where the potential is greater than the electron's total energy. The electron is reflected back with a partial tunnelling of the electron into the classically forbidden region.



Reflection and Transmission at Normal Incidence

Consider an EM wave incident normally upon a dielectric interface. There are two semi-infinite dielectrics, S and U, to the left and right in the figure below. Region S is the sourced region (there is a source of EM waves in this region).



The above diagram shows the incident, reflected and transmitted waves at normal incidence including the ***E*** and ***H*** vectors and **u** the unit vector in the direction of propagation. ***We note that all the fields are polarised in the plane of the interface i.e. with reference to our earlier discussion on boundary conditions, all of the fields are tangential to the interface and must therefore be continuous at the interface.***

The sourced medium has a dielectric constant, ε_S and unsourced medium a dielectric constant ε_U. Their electromagnetic impedances are $\eta_S = \sqrt{\frac{\mu_S \mu_0}{\varepsilon_S \varepsilon_0}}$ and $\eta_U = \sqrt{\frac{\mu_U \mu_0}{\varepsilon_U \varepsilon_0}}$ respectively. The distinction between the sourced and unsourced regions will later be important and should be noted.

Noting that the relation between ***E*** and ***H*** is through the impedance as

$$\eta H = E \tag{2.22}$$

Using the boundary conditions we have:

$$E_I + E_R = E_T \tag{2.23}$$

and

$$H_I - H_R = H_T \tag{2.24}$$

Relating the *H* fields to the *E* fields through the impedance we have

$$\frac{E_I}{\eta_S} - \frac{E_R}{\eta_S} = \frac{E_T}{\eta_U} \qquad (2.25)$$

rewritten

To find the amplitude transmission coefficient we can use 2.23 to eliminate E_R from 2.25 and we obtain

$$\frac{E_I}{\eta_S} - \frac{(E_T - E_I)}{\eta_S} = \frac{E_T}{\eta_U} \qquad (2.26)$$

Re arranging to get all E_I on one side of the equation and all E_T on the other

$$2E_I = \left[1 + \frac{\eta_S}{\eta_U}\right] E_T \qquad (2.27)$$

Re-arranging

$$E_T = \frac{2\eta_U}{\eta_S + \eta_U} E_I \qquad (2.28)$$

and

$$t = \frac{E_T}{E_I} = \frac{2\eta_U}{\eta_U + \eta_S} \qquad (2.29)$$

To find the amplitude reflection coefficient we can use 2.23 to eliminate E_T from 2.25 and we obtain

$$\frac{E_I}{\eta_S} - \frac{E_R}{\eta_S} = \frac{E_I + E_R}{\eta_U} \qquad (2.30)$$

Re arranging to get all E_I on one side of the equation and all E_R on the other

$$\frac{E_I}{\eta_S} - \frac{E_I}{\eta_U} = \frac{E_R}{\eta_U} + \frac{E_R}{\eta_S} \qquad (2.31)$$

$$\left(\frac{\eta_U - \eta_S}{\eta_U \eta_S}\right) E_I = \left(\frac{1}{\eta_U} + \frac{1}{\eta_S}\right) E_R$$

$$E_R = \frac{\eta_U - \eta_S}{\eta_U + \eta_S} E_I \tag{2.32}$$

and

$$r = \frac{E_R}{E_I} = \frac{\eta_U - \eta_S}{\eta_U + \eta_S} \tag{2.33}$$

Where t and r are the electric field amplitude transmission and reflection coefficients respectively.

We may note in passing the similarity between the result for r and the reflectivity of a voltage pulse on a transmission line of impedance Z_0 when terminated in a load impedance Z_L which is given by

$$r = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{2.34}$$

The coefficients may be written in terms of the refractive indices of the two media if ***they are not magnetic*** and $\mu_S = \mu_U = 1$

$$n_S = \sqrt{\varepsilon_S} \qquad\qquad \eta_S = \sqrt{\frac{\mu_0}{\varepsilon_S \varepsilon_0}} \tag{2.35}$$

Similarly for the unsourced medium.

$$n_U = \sqrt{\varepsilon_U} \qquad\qquad \eta_U = \sqrt{\frac{\mu_0}{\varepsilon_U \varepsilon_0}} \tag{2.36}$$

Therefore we can write

$$\eta_S = \frac{\eta_0}{n_S} \qquad \text{and} \qquad \eta_U = \frac{\eta_0}{n_U} \tag{2.37}$$

the reflection and transmission coefficients for non-magnetic media are therefore;

$$r = \frac{n_S - n_U}{n_S + n_U} \tag{2.38}$$

and

$$t = \frac{2n_S}{n_S + n_U} \tag{2.39}$$

NB

(i) The equations for reflection and transmission coefficients are not symmetric in either n nor η, i.e. it does matter which is the sourced region (as noted earlier). Eg. light incident normally on water from air will reflect to a differently to light incident normally on air from water. This is reflected in the expressions for r and t which are not symmetric wrt sourced and unsourced regions. However, we have found that the intensity, I, is proportional to the time average of the square of the field and the intensity reflection coefficient is $R = \left|r^2\right|$ and $T \propto \left|t^2\right|$ and looking at our expressions for r and t we can see that R will not depend on whether the light is reflected with air the sourced region or water. T will depend on such matters.

(ii) Also, all this was done for normal incidence where importantly all the fields were tangential to the interface and the boundary conditions derived earlier can be appled in a straightforward manner for all fields.

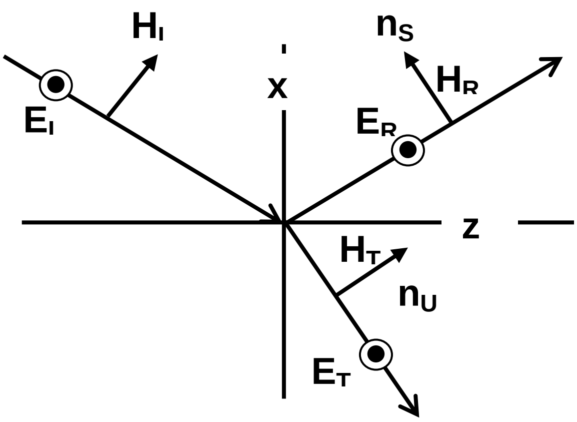
(iii) When writing in terms of n rather than η the subscripts in the expressions for r and t are interchanged in the new equations.

We have now obtained: reflection and transmission coefficients at normal incidence, Snell's law, the law of reflection, total internal reflection and evanescent fields. All have been derived from the simple boundary condition that ***tangential*** electric and magnetic fields match either side of a dielectric interface. We have only used tangential magnetic fields in the case of deriving the reflection and transmission coefficients.
Things get a little more tricky when deriving the reflection and transmission coefficients, r and t, for oblique incidence.

Reflection and Transmission at Oblique Incidence;

The Fresnel Equations.

The Fresnel equations are a set of equations that describe reflection and transmission coefficients for EM waves incident at an interface between two homogeneous, isotropic dielectrics, eg glass, water, for arbitrary angle of incidence.

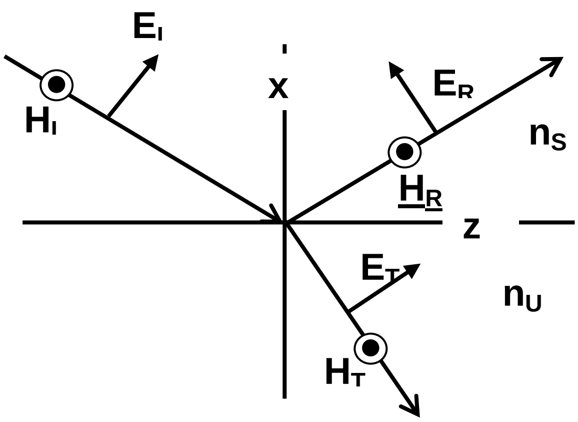


Transverse Electric, TE

$$E_I + E_R = E_T$$

$$H_I \cos\theta_I - H_R \cos\theta_I = H_T \cos\theta_T$$

$$\frac{E_I}{\eta_S}\cos\theta_I - \frac{E_R}{\eta_S}\cos\theta_I = \frac{E_T}{\eta_U}\cos\theta_T$$



Transverse Magnetic, TM

$$H_I + H_R = H_T$$

$$\frac{E_I}{\eta_S} + \frac{E_R}{\eta_S} = \frac{E_T}{\eta_U}$$

$$E_I \cos\theta_I - E_R \cos\theta_I = E_T \cos\theta_T$$

<u>TE and TM polarisations</u>

A wave of arbitrary polarisation will need to be resolved into components where the electric field vector is normal to the plane of incidence, *xz* plane in the above figures, (TE polarisation) and in the plane of incidence (TM polarisation). In constructing this diagram the right hand rule has been applied in determining the directions of *H* from direction of propagation and *E*. ie. index finger-*E*, middle finger-*H*, thumb-S(u or k). The magnetic field will be in the plane of incidence (TE polarisation) or normal to the plane of incidence (TM polarisation). These polarisations are often refered to in scientific literature as the **s polarisation (TE)** (from the German word for perpendicular, senkrecht) and **p polarisation (TM)** (from the German word for parallel, parallel) the polarisation of the electric field being perpendicular to and parallel to the plane of incidence respectively.

We make this distinction between the two polarisations because the boundary conditions require matching of the tangential components of the electric and magnetic fields. In the case of the TE polarisation the electric field is completely tangential to the interface whereas in the case of TM polarisation it is the magnetic field that is entirely tangential to the interface. The boundary conditions for *E* and *H* are also shown in the above diagrams.

We will now find the TE and TM reflection and transmission coefficients separately.

TE Polarisation.

At the boundary the BCs require:

$$E_{I_{Tan}} + E_{R_{Tan}} = E_{T_{Tan}} \qquad \Rightarrow \qquad E_I + E_R = E_T \tag{2.40}$$

Following the diagram (for the directions of the *H* fields) and using the BCs for H

$$H_{I_{Tan}} - H_{R_{Tan}} = H_{T_{Tan}} \qquad \Rightarrow H_I \cos\theta_I - H_R \cos\theta_I = H_T \cos\theta_T \tag{2.41}$$

We may as usual use the relation between *E* and *H* $H = \dfrac{E}{\eta}$ to obtain the *H* field boundary conditions in terms of *E*

$$\frac{E_I}{\eta_S}\cos\theta_I - \frac{E_R}{\eta_S}\cos\theta_I = \frac{E_T}{\eta_U}\cos\theta_T \tag{2.42}$$

We may now use equ 2.40 to eliminate E_T from 2.42

$$\frac{(E_I - E_R)}{\eta_S}\cos\theta_I = \frac{(E_I + E_R)}{\eta_U}\cos\theta_T \tag{2.43}$$

Grouping E_I and E_R on either side of a re-arranged equ 2.43

$$E_I(\eta_U \cos\theta_I - \eta_S \cos\theta_T) = E_R(\eta_U \cos\theta_I + \eta_S \cos\theta_T) \tag{2.44}$$

Finally we obtain the electric field amplitude reflection coefficient for the TE polarisation

$$r_E^{TE} = \frac{E_R}{E_I} = \frac{\eta_U \cos\theta_I - \eta_S \cos\theta_T}{\eta_U \cos\theta_I + \eta_S \cos\theta_T} \tag{2.45}$$

By dividing these Fresnel equations top and bottom by $\cos\theta_I \cos\theta_T$ we may obtain the Fresnel equations in a slightly different form as follows;

$$r_E^{TE} = \frac{\eta_U \sec\theta_T - \eta_S \sec\theta_I}{\eta_U \sec\theta_T + \eta_S \sec\theta_I} \tag{2.45a}$$

This is the first of four Fresnel equations.

The last operation (2.45 ⇒ 2.45a) was performed for reasons of symmetry, the angles associated with sourced and unsourced media are now compounded with the impedances associated with those media and we can compare 2.45a with the equivalent expression previously found for normal incidence, 2.33.

We may find the transmission coefficient in similar fashion for the TE polarisation using equation 2.40 to eliminate E_R in equ 2.42

$$E_R = E_T - E_I \tag{2.47}$$

$$\frac{1}{\eta_S}(2E_I - E_T)\cos\theta_I = \frac{1}{\eta_U} E_T \cos\theta_T \tag{2.48}$$

Grouping terms containing E_T and E_I

$$E_I\left[2\frac{1}{\eta_S}\cos\theta_I\right] = E_T\left[\frac{1}{\eta_S}\cos\theta_I + \frac{1}{\eta_U}\cos\theta_T\right] \tag{2.49}$$

Multiply both sides by $\eta_S \eta_U$

$$E_I[2\eta_U \cos\theta_I] = E_T[\eta_U \cos\theta_I + \eta_S \cos\theta_T]$$

We obtain the amplitude transmission coefficient for the TE polarisation

$$t_E^{TE} = \frac{E_T}{E_I} = \frac{2\eta_U \cos\theta_I}{\eta_U \cos\theta_I + \eta_S \cos\theta_T} \tag{2.50}$$

Again we dividing these Fresnel equations top and bottom by $\cos\theta_I \cos\theta_T$ to obtain the Fresnel equation in a more symmetric fashion as

$$t_E^{TE} = \frac{2\eta_U \sec\theta_T}{\eta_S \sec\theta_I + \eta_U \sec\theta_T} \tag{2.50a}$$

This is the second Fresnel equation. We now need to deal with the TM polarisation in a similar treatment to obtain the final two Fresnel equations.
NB. We can write the Fresnel equations in a number of different ways eg, 2.45 and 2.45a and 2.50 and 2.50a. Which we use is a matter of convenience, I have written 2.45a and 2.50a purely as a matter of aesthetics!

<u>TM Polarisation.</u>

Looking at the diagram for the TM polarised field to obtain the correct signs for the fields we find that continuity of the tangential E field at the interface now requires

$$E_I \cos\theta_I - E_R \cos\theta_I = E_T \cos\theta_T \tag{2.51}$$

For this polarisation the entire magnetic field is tangential to the interface thus the BC is

$$H_I + H_R = H_T \tag{2.52}$$

Writing 2.52 in terms of E

$$\frac{E_I}{\eta_S} + \frac{E_R}{\eta_S} = \frac{E_T}{\eta_U} \tag{2.53}$$

Use 2.53 to find E_T

$$E_T = \frac{\eta_U}{\eta_S}\left(E_I + E_R\right) \tag{2.54}$$

And eliminating E_T from 2.51

$$E_I \cos\theta_I - E_R \cos\theta_I = \frac{\eta_U}{\eta_S}\left(E_I + E_R\right)\cos\theta_T \qquad (2.55)$$

Grouping E_I and E_R either side of a re-arranged equ 2.55

$$E_I\left[\cos\theta_I - \frac{\eta_U}{\eta_S}\cos\theta_T\right] = E_R\left[\frac{\eta_U}{\eta_S}\cos\theta_T + \cos\theta_I\right] \qquad (2.56)$$

We thus obtain the electric field amplitude reflection coefficient for the TM polarisation

$$r_E^{TM} = \frac{E_R}{E_I} = \frac{\cos\theta_I - \frac{\eta_U}{\eta_S}\cos\theta_T}{\frac{\eta_U}{\eta_S}\cos\theta_T + \cos\theta_I} = \frac{\eta_S \cos\theta_I - \eta_U \cos\theta_T}{\eta_S \cos\theta_I + \eta_U \cos\theta_T} \qquad (2.57)$$

NB we could have found the reflection and transmission coefficients for the magnetic fields, H, and this might be considered more natural for the TM polarisation as the complete magnetic field is tangential to the boundar plane between the two dielectrics. The reflection coefficients would not be change for the magnetic field amplitudes as the reflected wave and incident wave are propagating in the same medium and *E* and *H* are related by the appropriate impedance.

$$r_M^{TM} = \frac{H_R}{H_I} = \frac{E_R/\eta_S}{E_I/\eta_S} = \frac{\eta_S \cos\theta_I - \eta_U \cos\theta_T}{\eta_S \cos\theta_I + \eta_U \cos\theta_T} \qquad (2.57a)$$

Finally by eliminating E_R from equ 2.51 using equ 2.53 rewritten as

$$E_R = \frac{\eta_S}{\eta_U} E_T - E_I$$

Substituting in 2.51

$$2E_I \cos\theta_I - \frac{\eta_S}{\eta_U} E_T \cos\theta_I = E_T \cos\theta_T \qquad (2.58)$$

Grouping the fields

$$2E_I \cos\theta_I = E_T\left[\frac{\eta_S}{\eta_U}\cos\theta_I + \cos\theta_T\right] \qquad (2.59)$$

Giving us the final Fresnel equation for the electric field amplitude coefficient of transmission for the TM polarisation

$$t_E^{TM} = \frac{E_T}{E_I} = \frac{2\cos\theta_I}{\frac{\eta_S}{\eta_U}\cos\theta_I + \cos\theta_T} = \frac{2\eta_U\cos\theta_I}{\eta_S\cos\theta_I + \eta_U\cos\theta_T} \tag{2.60}$$

To write the transmission coefficient in terms of the magnetic field requires account to be taken of the fact that the incident and transmitted waves travel in media of different impedance and

$$\left(\frac{H_T}{H_I}\right)^{TM} = t_M^{TM} = \frac{E_T/\eta_U}{E_I/\eta_S} = \frac{\eta_S}{\eta_U}\frac{E_T}{E_I} = \frac{2\eta_S\cos\theta_I}{\eta_S\cos\theta_I + \eta_U\cos\theta_T}$$

Equations 2.45, 2.50, 2.56 and 2.59 are the Fresnel equations.

THE FRESNEL EQUATIONS (Part i)

$$r_E^{TE} = \frac{\eta_U\sec\theta_T - \eta_S\sec\theta_I}{\eta_U\sec\theta_T + \eta_S\sec\theta_I}$$

$$t_E^{TE} = \frac{2\eta_U\sec\theta_T}{\eta_S\sec\theta_I + \eta_U\sec\theta_T}$$

$$r_E^{TM} = \frac{E_R}{E_I} = \frac{\eta_S\cos\theta_I - \eta_U\cos\theta_T}{\eta_S\cos\theta_I + \eta_U\cos\theta_T}$$

$$t_E^{TM} = \frac{E_T}{E_I} = \frac{2\eta_U\cos\theta_I}{\eta_S\cos\theta_I + \eta_U\cos\theta_T}$$

Also as most dielectrics in which we are interested are ***non-magnetic $\mu_1 = \mu_2 = 1$*** we may use the fact that $\eta_S = \frac{\eta_0}{n_S}$ and $\eta_U = \frac{\eta_0}{n_U}$ and write the Fresnel equations in terms of refractive indices rather than impedances as follows.

THE FRESNEL EQUATIONS (Part ii)

$$r_E^{TE} = \frac{n_S \cos\theta_I - n_U \cos\theta_T}{n_S \cos\theta_I + n_U \cos\theta_T} \qquad (2.60a)$$

$$t_E^{TE} = \frac{2n_S \cos\theta_I}{n_S \cos\theta_I + n_U \cos\theta_T} \qquad (2.60b)$$

$$r_E^{TM} = \frac{n_U \cos\theta_I - n_S \cos\theta_T}{n_U \cos\theta_I + n_S \cos\theta_T} \qquad (2.60c)$$

$$t_E^{TM} = \frac{2n_S \cos\theta_I}{n_U \cos\theta_I + n_S \cos\theta_T} \qquad (2.60d)$$

In the case of normal incidence where $\theta_I = \theta_T = 0$ there is no distinction between TE and TM and we can check this by verifying that we get the same reflection coefficient for TE and TM waves at normal incidence. At 0^0 the secant is 1 and the cosine is 1

$$r_E^{TE} = \frac{n_S \cos\theta_I - n_U \cos\theta_T}{n_S \cos\theta_I + n_U \cos\theta_T} = \frac{n_S - n_U}{n_S + n_U} \qquad (2.61a)$$

$$r_E^{TM} = \frac{n_U \cos\theta_I - n_S \cos\theta_T}{n_U \cos\theta_I + n_S \cos\theta_T} = \frac{n_U - n_S}{n_U + n_S} \qquad (2.61b)$$

Thus we have identical absolute magnitudes for the amplitude reflection coefficients but a change in sign. This should concern us as the TE and TM waves are equivalent in this case. The problem occurs at the start of this section on page 36 where the TE and TM waves were drawn in order to obtain the boundary conditions of equs 2.29 and 2.40. In constructing those figures the assumptions were that the right hand rule applies in determining the directions of *H* from direction of propagation and *E*. ie. right hand, index finger-*E*, middle finger-*H*, thumb-*S*(*u* or *k*). This procedure is of course correct in principle but something else occurs at the interface which we have ignored.

We made a further assumption that the electric field doesn't change direction in the reflected wave for TM. In fact the E field reverses upon reflection. This implies a phase change upon reflection which we have ignored. It also alters the boundary conditions. In fact, the technique is correct and as we want the intensity reflection coefficients the phase change plays no role. To find the intensity reflection coefficients we will square the amplitude reflection coefficient and therefore the sign is irrelevant.

We can obtain the Fresnels equations in even more compact form starting with the expression for r_E^{TE} as follows:

$$r_E^{TE} = \frac{n_S \cos\theta_I - n_U \cos\theta_T}{n_S \cos\theta_I + n_U \cos\theta_T}$$

Application of Snell's Law gives us

$$n_S = n_U \frac{\sin\theta_T}{\sin\theta_I}$$

We can substitute into the equation for r_E^{TE}

$$r_E^{TE} = \frac{n_U\left(\dfrac{\sin\theta_T \cos\theta_I}{\sin\theta_I} - \cos\theta_T\right)}{n_U\left(\dfrac{\sin\theta_T \cos\theta_I}{\sin\theta_I} + \cos\theta_T\right)}$$

Multiplying top and botom by $\sin\theta_I$ leaves

$$r_E^{TE} = \frac{\sin\theta_T \cos\theta_I - \sin\theta_I \cos\theta_T}{\sin\theta_T \cos\theta_I + \sin\theta_I \cos\theta_T} = \frac{\sin(\theta_T - \theta_I)}{\sin(\theta_T + \theta_I)} = -\frac{\sin(\theta_I - \theta_T)}{\sin(\theta_I + \theta_T)}$$

And for t_E^{TE} we can similarly compactify thus;

$$t_E^{TE} = \frac{2 n_S \cos\theta_I}{n_S \cos\theta_I + n_U \cos\theta_T}$$

From using Snells law

$$n_S = n_U \frac{\sin\theta_T}{\sin\theta_I}$$

$$t_E^{TE} = \frac{2n_U \,{}^{sin\theta_T}\!\big/_{sin\theta_I}\, cos\,\theta_I}{n_U\left({}^{sin\theta_T}\!\big/_{sin\theta_I}\, cos\,\theta_I + cos\,\theta_T \right)}$$

Multiply top and bottom by $sin\theta_I$

$$t_E^{TE} = \frac{2\,sin\,\theta_T\, cos\,\theta_I}{(sin\,\theta_I\, cos\,\theta_I + sin\,\theta_I\, cos\,\theta_T)}$$

Using a trigonometric identity this may be simplified to

$$t_E^{TE} = \frac{2\,sin\,\theta_T\, cos\,\theta_I}{sin(\,\theta_I + \theta_T\,)}$$

We now compact the equivalent expressions for the Fresnel equations of the TM polarisation;

$$r_E^{TM} = \frac{n_U\, cos\,\theta_I - n_S\, cos\,\theta_T}{n_U\, cos\,\theta_I + n_S\, cos\,\theta_T}$$

Use Snells law to replace n_U

$$r_E^{TM} = \frac{n_S\left({}^{sin\theta_I}\!\big/_{sin\theta_T}\, cos\,\theta_I - cos\,\theta_T \right)}{n_S\left({}^{sin\theta_I}\!\big/_{sin\theta_T}\, cos\,\theta_I + cos\,\theta_T \right)}$$

Multiply top and bottom by $sin\theta_T$

$$r_E^{TM} = \frac{sin\,\theta_I\, cos\,\theta_I - sin\,\theta_T\, cos\,\theta_T}{sin\,\theta_I\, cos\,\theta_I + sin\,\theta_T\, cos\,\theta_T}$$

Using trigonometric identities

$$r_E^{TM} = \frac{sin(\,\theta_I - \theta_T\,)}{cos(\,\theta_I - \theta_T\,)}\frac{cos(\,\theta_I + \theta_T\,)}{sin(\,\theta_I + \theta_T\,)}$$

To obtain the final compact equation

$$r_E^{TM} = \frac{tan(\theta_I - \theta_T)}{tan(\theta_I + \theta_T)}$$

And finally we rewrite the transmission coefficient for the TM wave;

$$t_E^{TM} = \frac{2n_S \cos\theta_I}{n_U \cos\theta_I + n_S \cos\theta_T}$$

Using Snells law to replace n_U

$$n_U = n_S \frac{\sin\theta_I}{\sin\theta_T}$$

$$t_E^{TM} = \frac{2n_S \cos\theta_I}{n_S\left(\sin\theta_I / \sin\theta_T \cos\theta_I + \cos\theta_T\right)}$$

Multiply top and bottom by $\sin\theta_I$

$$t_E^{TM} = \frac{2\sin\theta_I \cos\theta_I}{\sin^2\theta_I / \sin\theta_T \cos\theta_I + \sin\theta_I \cos\theta_T}$$

$$t_E^{TM} = \frac{2\sin\theta_I \cos\theta_I}{\left(\sin^2\theta_I \cos\theta_I + \sin\theta_T \sin\theta_I \cos\theta_T\right) / \sin\theta_T}$$

$$t_E^{TM} = \frac{2\sin\theta_I \sin\theta_T \cos\theta_I}{\sin\theta_I\left(\sin\theta_I \cos\theta_I + \sin\theta_T \cos\theta_T\right)}$$

And finally

$$t_E^{TM} = \frac{2\sin\theta_T \cos\theta_I}{\sin(\theta_I + \theta_T)\cos(\theta_I - \theta_T)}$$

We may group the four compact Fresnel equations together

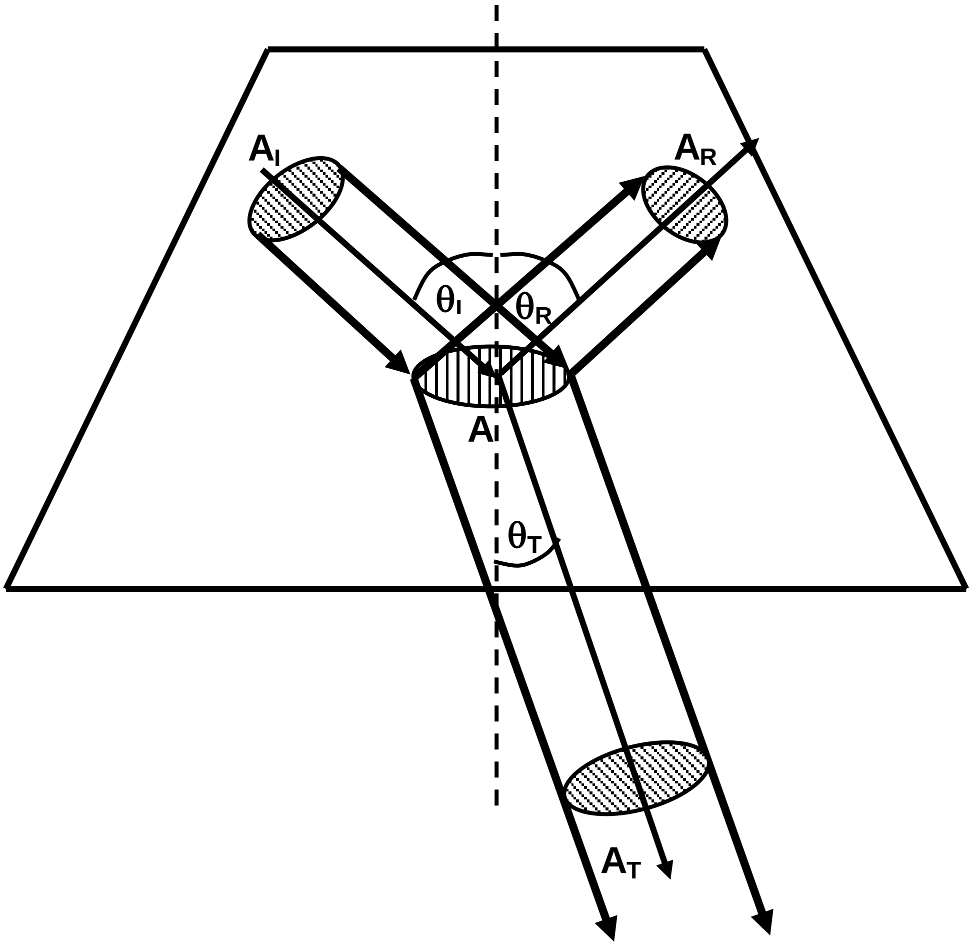
THE FRESNEL EQUATIONS (Part iii)

$$r_E^{TE} = -\frac{sin(\theta_I - \theta_T)}{sin(\theta_I + \theta_T)}$$

$$t_E^{TE} = \frac{2 sin\theta_T cos\theta_I}{sin(\theta_I + \theta_T)}$$

$$r_E^{TM} = \frac{tan(\theta_I - \theta_T)}{tan(\theta_I + \theta_T)}$$

$$t_E^{TM} = \frac{2 sin\theta_T cos\theta_I}{sin(\theta_I + \theta_T)cos(\theta_I - \theta_T)}$$



To obtain *intensity* reflection coefficients and transmission coefficients requires a bit more work. So far we have been interested in the electric field or amplitude reflection coefficients. As we have already learned the intensity is related to the electric field by

$$I = \frac{E^2}{2\eta} = \frac{1}{2}\sqrt{\frac{\varepsilon\varepsilon_0}{\mu\mu_0}}E^2 = \frac{1}{2}n\sqrt{\frac{\varepsilon_0}{\mu_0}}E^2 = \frac{1}{2}\frac{n}{\eta_0}E^2 \tag{2.62}$$

For the reflection coefficients the situation is simple as the reflected and incident waves are in the same medium and *n* has the same value when converting E_I and E_R and therefore the reflection coefficient is simply

$$R = \frac{P_R}{P_I} = \frac{I_R \times A_R}{I_I \times A_I} = \frac{E_R^2 \big/ 2\eta_S \; A_R}{E_I^2 \big/ 2\eta_S \; A_R} = r^2 \tag{2.63}$$

for both TE and TM.

$$R_E^{TE} = \left[\frac{n_S\, cos\,\theta_I - n_U\, cos\,\theta_T}{n_S\, cos\,\theta_T + n_U\, cos\,\theta_I}\right]^2 \qquad (2.64a)$$

$$R_E^{TM} = \left[\frac{n_U\, cos\,\theta_I - n_S\, cos\,\theta_T}{n_U\, cos\,\theta_I + n_S\, cos\,\theta_T}\right]^2 \qquad (2.64c)$$

For the transmission coefficients the incident and transmitted waves are travelling in different media and the refractive indices will differ. This must be accounted for along with the fact that it is the power flow normal to the direction of propagation that we are interested in. This introduces a cosine factor as the area normal to the direction of propagation

$$A_{Norm} = \vec{A} \bullet \hat{S} = \vec{A} \bullet \hat{k}_i = A\cos\theta \qquad (2.65)$$

is altered by this factor. A is the area on the interface, $\hat{k}$ and $\hat{S}$ are the unit wavevector and Poynting vector respectively.

Using $t_E^{TE}, t_E^{TM} \equiv \frac{E_T}{E_I}$

$$T_E^{TE}, T_E^{TM} = \frac{P_T}{P_I} = \frac{I_T A_T}{I_I A_I} = \frac{I_T A\cos\theta_T}{I_I A\cos\theta_I} = \frac{E_T^2}{E_I^2}\frac{n_U}{n_S}\frac{\cos\theta_T}{\cos\theta_I} \qquad (2.66)$$

Using the expression for t_{TE} from equ 2.50b in 2.56

$$T_E^{TE} = \left[\frac{2n_S\, cos\,\theta_I}{n_S\, cos\,\theta_I + n_U\, cos\,\theta_T}\right]^2 \frac{n_U}{n_S}\frac{cos\,\theta_T}{cos\,\theta_I} \qquad (2.66b)$$

Using the expression for t_{TM} from equ 2.50d in 2.56

$$T_E^{TM} = \left[\frac{2n_S \cos\theta_I}{n_U \cos\theta_I + n_S \cos\theta_T}\right]^2 \frac{n_U}{n_S}\frac{\cos\theta_T}{\cos\theta_I} \qquad (2.66d)$$

We should note that all of the Fresnel equations, as shown, involve both the incident angle, θ_I , and the transmitted angle, θ_T. To obtain reflection and transmission coefficients in terms of θ_I alone it would be necessary to use Snell's law to eliminate θ_T.

Conservation of Energy

To conserve energy at the boundary we are considering, the power incident must be , in the absence of absorption, equal to the sum of the power reflected and the power transmitted

$$P_I = P_R + P_T \qquad (2.68)$$

Using the definitions of the reflectance and transmittance

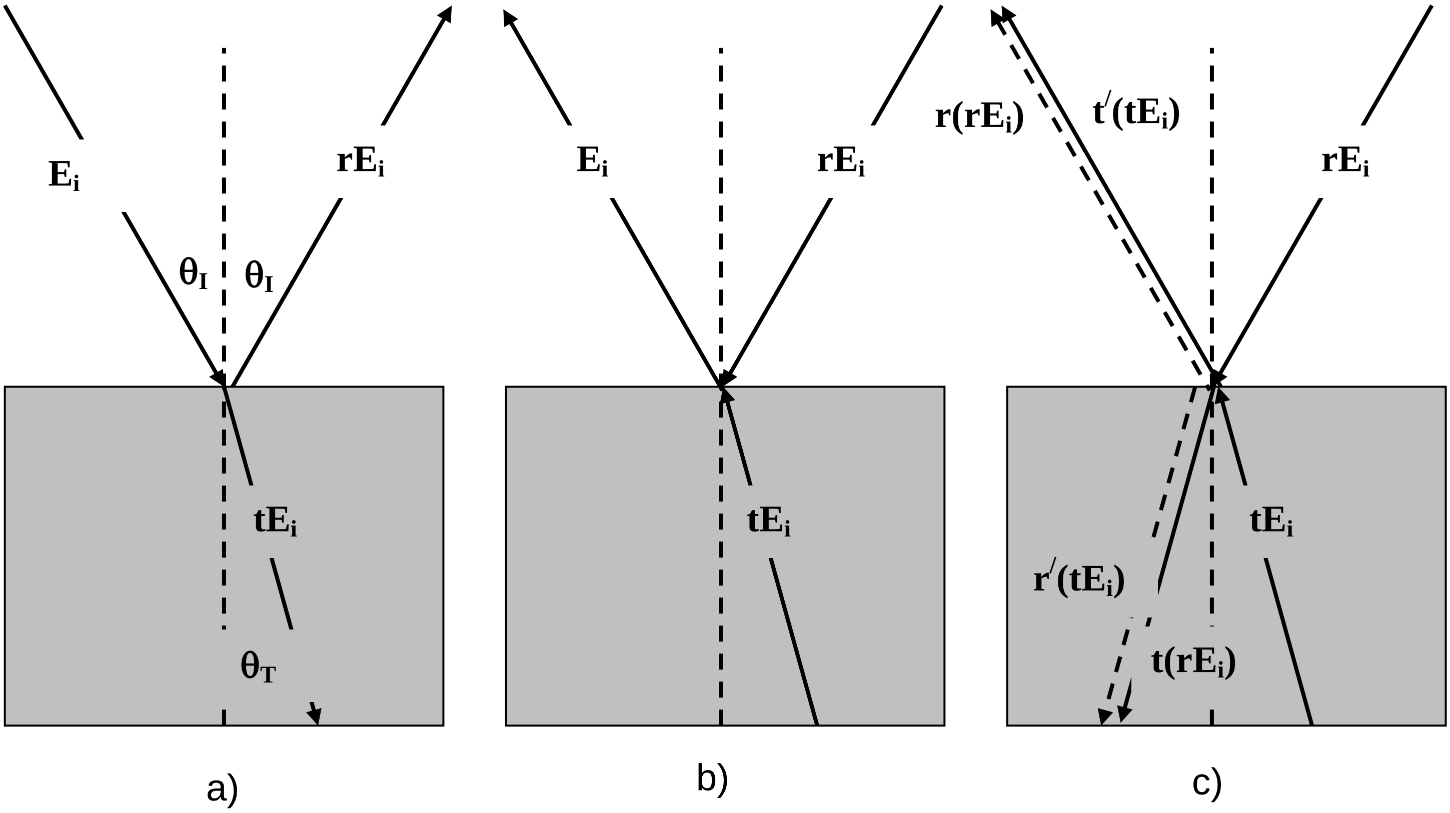
$$R = \frac{P_R}{P_I} \qquad T = \frac{P_T}{P_I} \qquad (2.69)$$

therefore

$$1 = R + T \qquad (2.70)$$

The Stokes Relations

Fermat's principle of least action requires that, ***in the absence of dissipation*** in the form of absorption, ***any viable ray diagram*** (such as the usual diagram shown in figure a) of a ray incident at a dielectric interface undergoing refraction and reflection) ***will also work with all ray directions reversed*** as in figure b). An argument due to Stokes invokes Fermat to demonstrate a relation between the amplitude transmission and reflection coefficients, r and t.



Clearly, in figure b) there is an internal reflection missing. Also the reflected ray in the upper half of the diagram should be a composite of a refracted ray from the incident ray in the lower half and the incident ray in the upper half. These component rays have been added in figure c). Figures b) and c) must be equivalent.

If the configuration in figures c) and b) are equivalent then it follows that

$$tt^{/}E_i + rrE_i = E_i \tag{2.71}$$

and

$$rtE_i + tr^{/}E_i = 0 \tag{2.72}$$

From 2.71

$$tt^{/} = 1 - r^2 \tag{2.73}$$

From 2.72

$$r = -r^{/} \tag{2.74}$$

We need to be a bit circumspect and note that the amplitude reflection and transmission coefficients depend on angle and the equations 2.73 and 2.74 need to include this fact explicitly and so are written

$$r(\theta_I) = -r^{/}(\theta_T) \tag{2.73a}$$

$$t(\theta_I)t^{/}(\theta_T) = 1 - r^2(\theta_I) \tag{2.74a}$$

Equations 2.73 and 2.74 are known as the **Stokes relations**

Of course we bear in mind that θ_I and θ_T always come as a pair of angles related through Snell's law.

Brewster's Angle

One consequence of the tunability (by angle) of the impedance of a dielectric medium is that ***it is possible in certain circumstances*** to find an angle where the impedance of the unsourced medium matches that of the sourced medium and at that angle the reflectivity becomes zero.

For TM waves, the angle at which r becomes zero is known as **Brewster's angle.**

We can find this angle by considering the Fresnel equation for amplitude reflection coefficient

$$r_M^{TM} = \frac{\eta_U \cos\theta_T - \eta_S \cos\theta_I}{\eta_U \cos\theta_T + \eta_S \cos\theta_I}$$

In non-magnetic media this may be rewritten;

$$r_M^{TM} = \frac{n_S \cos\theta_T - n_U \cos\theta_I}{n_S \cos\theta_T + n_U \cos\theta_I}$$

We see that $r_M^{TM} = 0$ when $n_S \cos\theta_T = n_U \cos\theta_I$

i.e. at some angle of incidence when $\theta_I = \theta_B$ the reflectivity becomes zero.

Re-arranging we have

$$\cos^2\theta_T = \left(\frac{n_U}{n_S}\right)^2 \cos^2\theta_B \tag{2.75}$$

From Snell's law

$$\sin\theta_T = \frac{n_S}{n_U}\sin\theta_B$$

Now we can use a trigonometric identity to write

$$\cos^2\theta_T + \sin^2\theta_T = 1 = \left(\frac{n_S}{n_U}\right)^2 \sin^2\theta_B + \left(\frac{n_U}{n_S}\right)^2 \cos^2\theta_B = \cos^2\theta_B + \sin^2\theta_B \tag{2.76}$$

Collecting sin and cos terms together

$$\left[\left(\frac{n_S}{n_U}\right)^2 - 1\right] sin^2\,\theta_B = \left[1 - \left(\frac{n_U}{n_S}\right)^2\right] cos^2\,\theta_B \tag{2.77}$$

$$tan^2\,\theta_B = \frac{1 - \left(n_U / n_S\right)^2}{\left(n_S / n_U\right)^2 - 1} = \frac{\dfrac{n_S^2 - n_U^2}{n_S^2}}{\dfrac{n_S^2 - n_U^2}{n_U^2}} = \frac{n_U^2}{n_S^2} \tag{2.78}$$

The Brewster angle for TM waves is given by

$$tan\,\theta_B = \frac{n_U}{n_S} \tag{2.79}$$

When $\theta_I = \theta_B$ for a TM wave then $r_M^{TM} = 0 = r_E^{TM}$

NB. It is important that the equation is used the correct way around with the refractive index of the unsourced region appearing in the numerator! This is another example of the importance of noting the difference between the sourced and unsourced regions when using the equations derived here.

For TE waves we may follow the same methodology

$$\eta_U\, sec\,\theta_T = \eta_S\, sec\,\theta_I \qquad \text{or} \qquad \eta_U\, cos\,\theta_I = \eta_S\, cos\,\theta_T \tag{2.80}$$

For $\theta_I = \theta_B$

$$n_S\, cos\,\theta_B = n_U\, cos\,\theta_T$$

For $r_E^{TE} = 0$

Re-written

$$\frac{n_S}{n_U} cos\,\theta_B = cos\,\theta_T \tag{2.81}$$

Following the same procedure

$$\cos^2\theta_T+\sin^2\theta_T=1=\left(\frac{n_S}{n_U}\right)^2\sin^2\theta_B+\left(\frac{n_S}{n_U}\right)^2\cos^2\theta_B=\cos^2\theta_B+\sin^2\theta_B \qquad (2.82)$$

Again collecting sin and cos terms

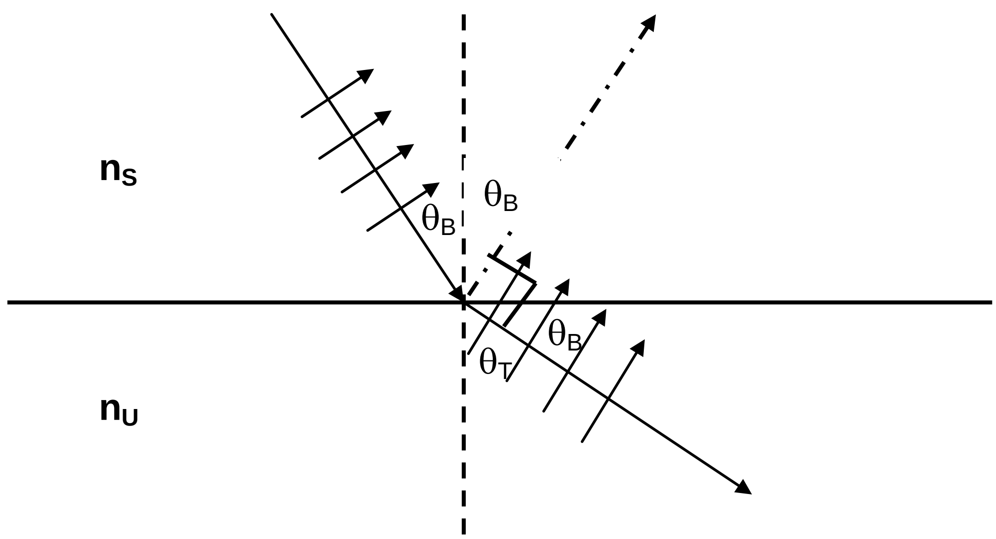
$$\left[\left(\frac{n_S}{n_U}\right)^2-1\right]\sin^2\theta_B=\left[1-\left(\frac{n_S}{n_U}\right)^2\right]\cos^2\theta_B \qquad (2.83)$$

$$\tan^2\theta_B=\frac{1-\left({n_S}/{n_U}\right)^2}{\left({n_S}/{n_U}\right)^2-1}=\frac{\dfrac{n_U^2-n_S^2}{n_U^2}}{\dfrac{n_S^2-n_U^2}{n_U^2}}=-1 \qquad (2.84)$$

While $\tan\theta_B$ may take on any value from 0 to ∞ this would require $\tan\theta_B$ to be imaginary!

Clearly the condition for $r_E^{TE}=0$ cannot be satisfied and ***there is NO Brewster angle for the TE polarisation.***

We are able to deduce the expression for the Brewster angle geometrically as follows



Recalling that the polarisation is perpendicular to the direction of propagation and that the E polarisation of the medium gives rise to the secondary waves at that polarisation that are the reflected waves, we can see that when the refracted ray is ***at right angles to the direction the reflected wave should propagate in*** according to laws of reflection then for TM waves there can be no reflected wave in the case of TM waves.

The polarisation induced by the incident wave would lie in the same direction as the reflected wave which is not allowed. Therefore at this particular angle of incidence where the reflected ray would be at right angles to the transmitted ray there is zero reflection coefficient. This is therefore the Brewster angle.

We find the Brewster angle as follows from the ray diagram

$$n_S \sin\theta_B = n_U \sin\theta_T = n_U \sin(90-\theta_B) = n_U \cos\theta_B$$

$$\tan\theta_B = \frac{n_U}{n_S}$$

This is the same as the answer we found previously.
NB.

1) The result gives the tangent of the Brewster angle as

index of the unsourced region divided by **index of the sourced region.**

2) The Brewster angle exists only for the TM wave a similar angle cannot be described for a TE wave. It wil always be possible to derive a secondary (reflected) wave from the polarisations induced by a TE wave without going against any fundamental principle.

3) The Brewster angle exists irrespective of the relative size of the refractive indices (unlike the critical angle which exists only when $n_S > n_U$).

We note (as is evident in the construction) that for $n_U > n_S$, θ_B is greater than 45^0 and for $n_U < n_S$, θ_B is less than 45^0.

The Brewster angle is used to produce polarising elements in laser tubes where Brewster windows allow TE polarised waves to suffer greater losses (due to reflection) than TM waves.

Highlights of Part 2

Law of Reflection $\theta_I = \theta_R$

Law of Refraction (Snell's Law) $n_S \sin\theta_I = n_U \sin\theta_T$ ***

Critical Angle $\theta_C = \sin^{-1}\dfrac{n_S}{n_U}$ $\quad n_S > n_U$

Brewster Angle $\theta_B = \tan^{-1}\dfrac{n_U}{n_S}$

Evanescent wavevector is $\alpha = \sqrt{k_1^2 \sin^2\theta_I - k_2^2} = k_0\sqrt{n_1^2 \sin^2\theta_I - n_2^2}$

Reflection coeff, normal incidence $r = \dfrac{E_R}{E_I} = \dfrac{\eta_U - \eta_S}{\eta_U + \eta_S} = \dfrac{n_S - n_U}{n_S + n_U}$

Transmission coeff, normal incidence $t = \dfrac{E_T}{E_I} = \dfrac{2\eta_U}{\eta_U + \eta_S} = \dfrac{2n_S}{n_S + n_U}$

***** The subscripts S and U used here refer to sourced and unsourced regions respectively.**

The subscripts I , R and T refer to incidence, reflection and transmission angles respectively.

Highlights of Part 2 cont.

The Fresnel equations (non magnetic)

The amplitude coefficients

$$r_E^{TE} = \frac{E_R}{E_I} = \frac{n_S \cos\theta_I - n_U \cos\theta_T}{n_S \cos\theta_I + n_U \cos\theta_T}$$

$$t_E^{TE} = \frac{E_T}{E_I} = \frac{2n_S \cos\theta_I}{n_S \cos\theta_I + n_U \cos\theta_T}$$

$$r_E^{TM} = \frac{E_R}{E_I} = \frac{n_U \cos\theta_I - n_S \cos\theta_T}{n_U \cos\theta_I + n_S \cos\theta_T}$$

$$t_E^{TM} = \frac{E_T}{E_I} = \frac{2n_S \cos\theta_I}{n_U \cos\theta_I + n_S \cos\theta_T}$$

The Fresnel equations (non magnetic)

The intensity coefficients

$$R_{TE} = \left[\frac{n_S \cos\theta_I - n_U \cos\theta_T}{n_S \cos\theta_T + n_U \cos\theta_I}\right]^2$$

$$T_{TE} = \left[\frac{2n_S \cos\theta_I}{n_S \cos\theta_I + n_U \cos\theta_T}\right]^2 \frac{n_U}{n_S} \frac{\cos\theta_T}{\cos\theta_I}$$

$$R^{TM} = \left[\frac{n_U \cos\theta_I - n_S \cos\theta_T}{n_U \cos\theta_I + n_S \cos\theta_T}\right]^2$$

$$T^{TM} = \left[\frac{2n_S \cos\theta_I}{n_U \cos\theta_I + n_S \cos\theta_T}\right]^2 \frac{n_U}{n_S} \frac{\cos\theta_T}{\cos\theta_I}$$