4. POLARISATION.

Plane polarised light

Earlier, we considered one of the most useful solutions to the electromagnetic wave equation, the plane wave, travelling in the z direction, say. That is a transverse wave whose constant phase fronts are planes perpendicular to z, ie. the xy plane. We write the plane wave in one of two forms;

$$\vec{E}(z,t) = \vec{A}_0 \cos\left[2\pi\left(\frac{z}{\lambda} - \nu t\right)\right] = \vec{A}_0 \cos[kz - \omega t] \qquad (4.1a)$$

or equivalently

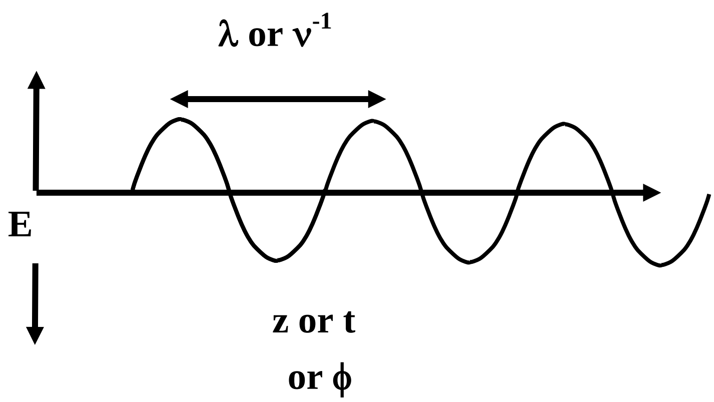
$$\vec{E}(z,t) = \vec{A}_0 \exp j[kz - \omega t] \qquad (4.1b)$$

We noted one important property of a wave, its phase and in the above case the phase is simply the argument of the cosinusoid or exponential,

$$\phi = kz - \omega t$$

and this phase does not depend on x or y once z has been fixed, ie. the phase is the same at any value of x and y for a given value of z and t.

The plane wave is usually represented with a diagram such as that shown below where horizontally either position (time constant) or time (position constant) is plotted and vertically the ***magnitude of the electric field*** is plotted.



So far we have ignored one very important property of the transverse plane wave. In equations 4.1 the amplitude of the wave is correctly written as a vector, $\vec{A}_0$. ie. the electric field possesses not only magnitude but direction. The vector $\vec{E}$ will be lying somewhere in the xy plane for a wave propagating in the z direction ie. it has a polarisation. Furthermore, in general that direction may alter with time or position. In general a light source such as the sun or an incandescent light bulb continuously emits separate uncorrelated wave trains (in quantum transitions) that are independent of one another and the light produced is overall unpolarised.

NB. For the light to acquire an overall polarisation the involvement of some polarising element or process is required.

eg. It has already been noted in discussions around the Fresnel equations that the reflectivity or transmissivity depends on the polarisation of the light and two types of plane polarisation were introduced in that discussion in order to arrive at the equations, that is the Transverse Electric or TE polarisation and the Transverse Magnetic or TM polarisation which are mutually orthogonal and defined with respect to an interface (plane of incidence) between two dielectrics. It was noted that in an extreme case, at the Brewster angle the reflectivity of the TM polarisation went to zero leaving only the TE polarisation in the reflected wave. A dielectric orientated at the Brewster angle is an example of a polarising element. This is also the reason why light reflected from the surface of water at certain angles is polarised.

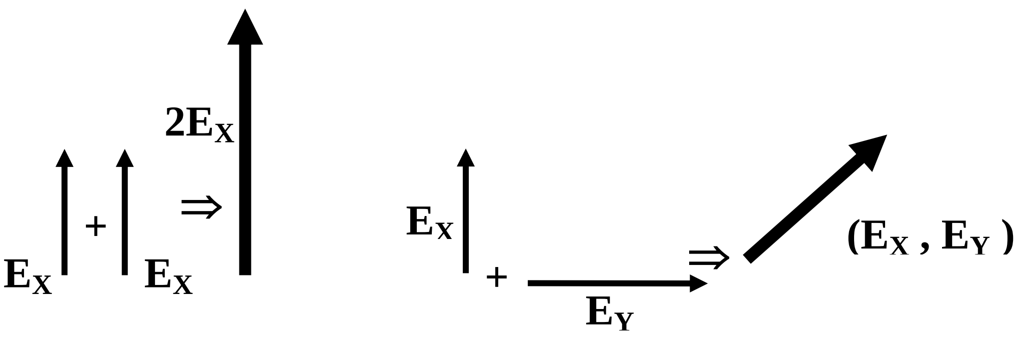
In some special circumstances the polarisation state can be specified. We shall list these beginning with the simplest case.

1. Plane Polarised electromagnetic waves where ***the polarisation direction is uniquely specified and is independent of time and position***.

For a wave travelling in a direction **k** the plane of polarisation is that defined by **k** and **E**.

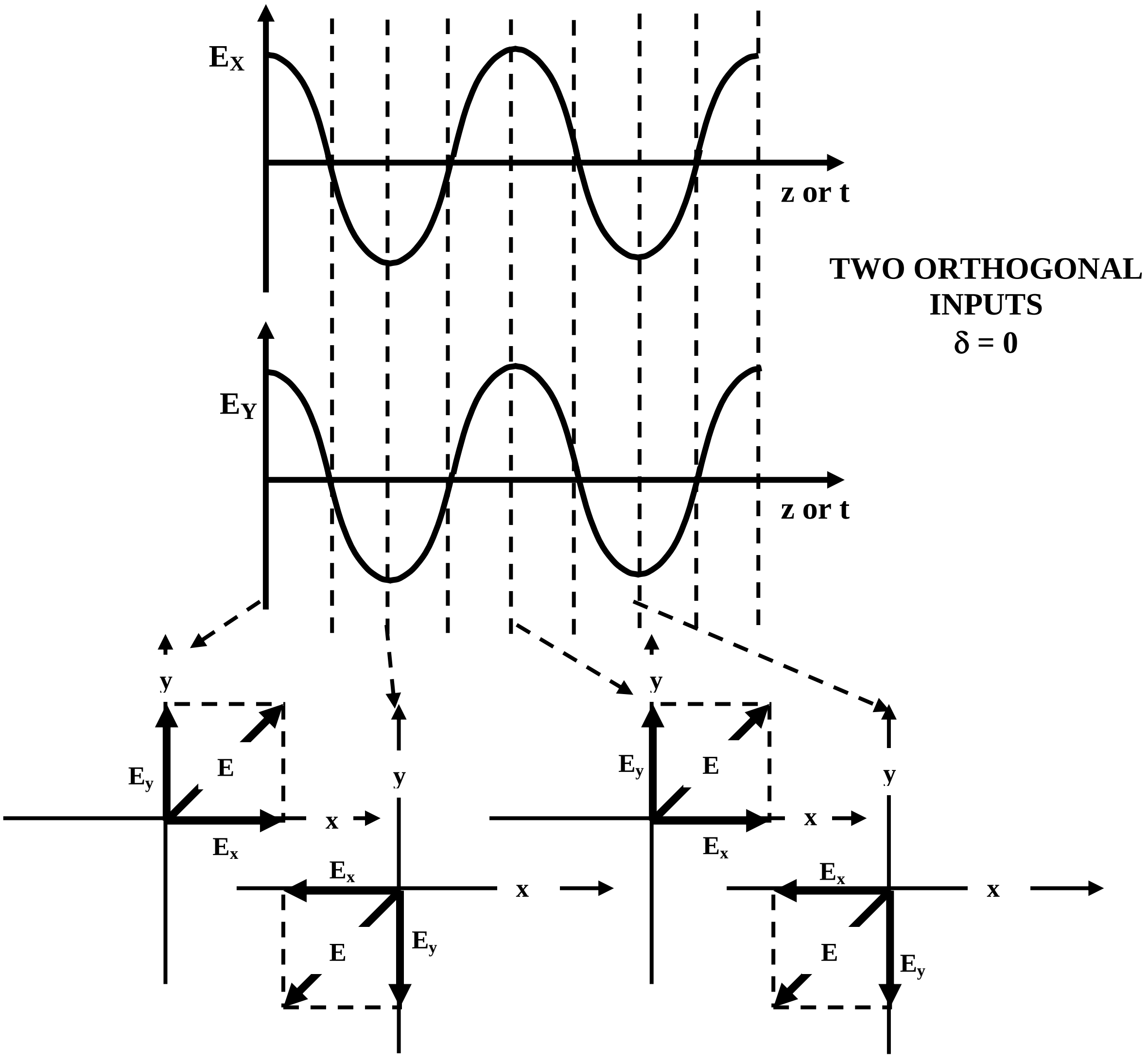
Recalling from earlier electromagnetism courses the superposition principle which states that if electric fields from two or more sources are present in the same region of space at the same time there is a net field that is found by the simple addition of the fields as vectors.

We have several simple situations of **electrostatic fields** to consider.



The examples shown are of static fields where the two fields may or may not be in the same direction. This is simple vector addition but needs to be re-emphasised before continuing to look at further aspects of polarisation and of examining the superposition principle in general and how it determines the effects of interference and diffraction.

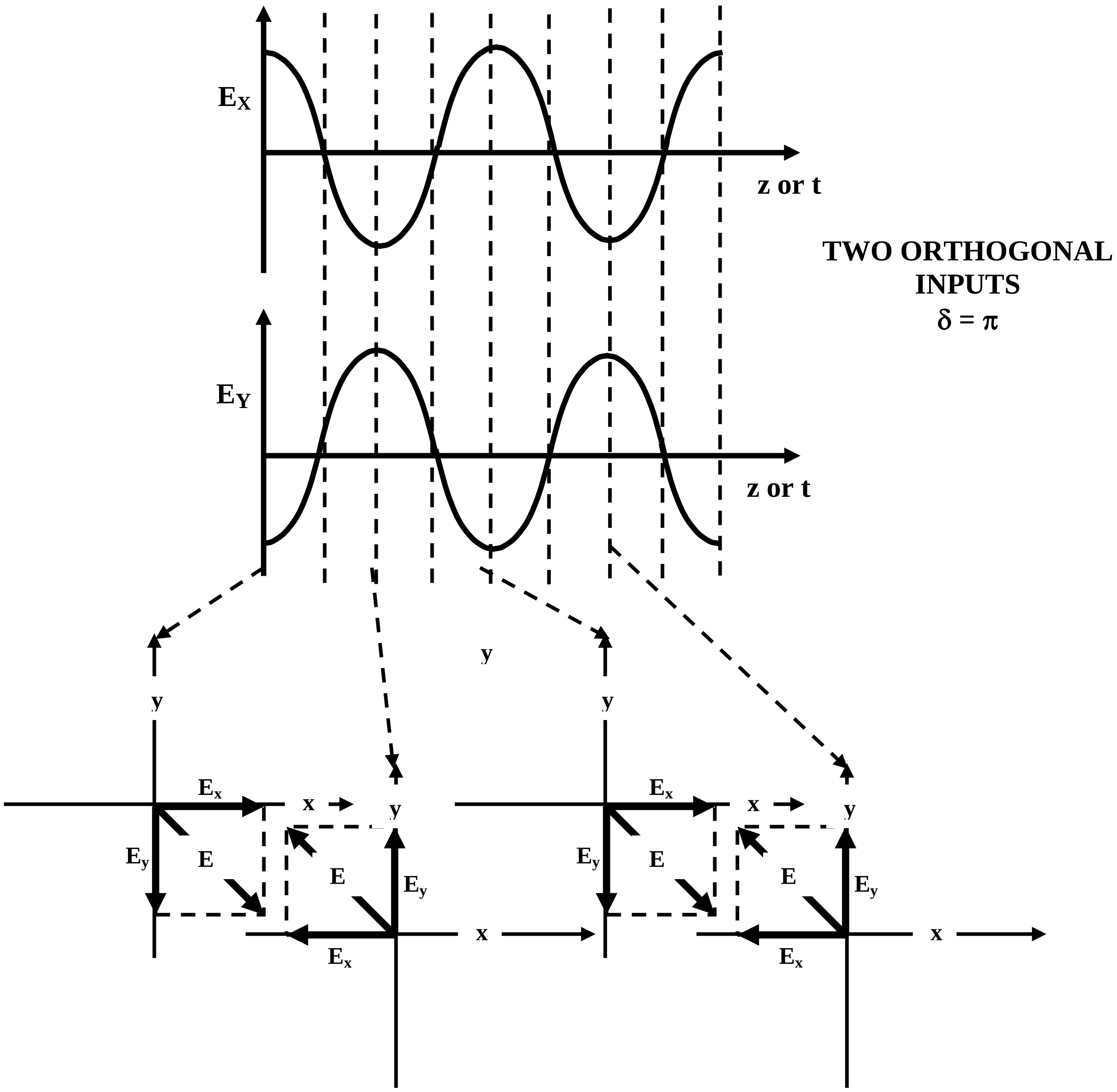
Of more interest is to do the same vector additions with electric fields that vary in space or in time as a plane wave.



The first example to analyse is the case of two electromagnetic waves of the same amplitude, frequency and wavelength propagating in the same direction, say z and **propagating in phase**. One of the waves is plane polarised in the x direction and the other plane polarised in the y direction. The situation is depicted in the above diagrams along with the resultant field worked out at the peaks and troughs. We find that the resultant is again a plane polarised wave whose polarisation direction is oscillating at $+45^0$ to the x and y axes. The amplitude of the resultant is

$$E = \sqrt{E_x^2 + E_y^2} = \sqrt{2}E_x \tag{4.2}$$

If the amplitude of the two fields had been different then the wave would have remained plane polarised but at an angle $\theta = \tan^{-1}\dfrac{E_y}{E_x}$ to the x axis.



RESULTANT

Another example, depicted above, similar to the first is that of two electromagnetic waves of the same amplitude, frequency and wavelength but of mutually orthogonal plane polarisation propagating in the same direction but with a phase lag $\delta = \pi = 180^0$ with respect to one another. The resultant field is shown again at the peaks and troughs. This is another plane polarised wave with its polarisation vector at -45^0 to the x axis. We can also see this mathematically;

In the first case with zero phase shift

$$\left.\begin{aligned} E_x &= E_{0x}\hat{x}\cos(kz-\omega t) \\ E_y &= E_{0y}\hat{y}\cos(kz-\omega t) \\ E &= (\hat{x}+\hat{y})E_0\cos(kz-\omega t) \end{aligned}\right\} \quad (4.3)$$

<u>**NB. In Hecht the notation ϕ is used for the phase lag, ie. ϕ ≡ δ, here I use ϕ for the phase generally.**</u>

This is a field where the vector is $(\hat{x}+\hat{y})$ pointing in the direction at 45^0 to the *x* (or *y*) axis oscillating with a frequency ω and a wavelength $\lambda = \frac{2\pi}{k}$. Note that the vector $(\hat{x}+\hat{y})$ pointing in a direction +45^0 to the x axis is no longer a unit vector of magnitude 1 but has magnitude $\sqrt{1^2+1^2} = \sqrt{2}$ and therefore the amplitude of the resultant field in 4.3 is $\sqrt{2}E_0$ where $E_0 = E_{0x} = E_{0y}$ as stated.

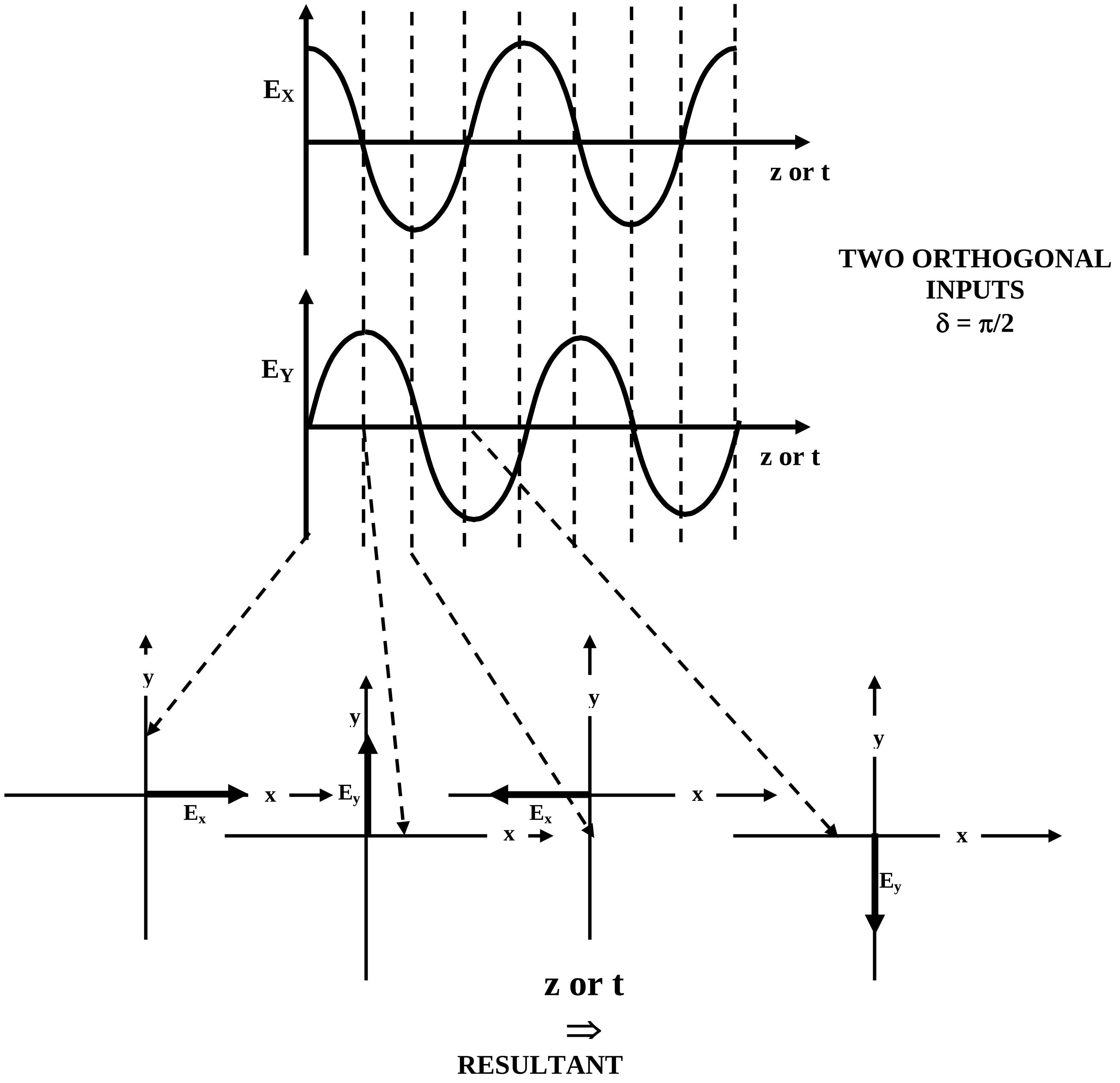
In the second case with phase shift $\delta = \pi = 180^0$ between x and y electric field components

$$\left.\begin{aligned} E_x &= E_{0x}\hat{x}\cos(kz-\omega t) \\ E_y &= E_{0y}\hat{y}\cos(kz-\omega t+\delta) = E_{0y}\hat{y}\cos(kz-\omega t+\pi) = -E_0\hat{y}\cos(kz-\omega t)) \\ E &= (\hat{x}-\hat{y})E_0\cos(kz-\omega t) \end{aligned}\right\} \quad (4.4)$$

In this case the field vector is $(\hat{x}-\hat{y})$ and points in the direction at -45^0 to the x axis.

It is not inevitably the case that the two orthogonal plane polarised and co-propagating electric fields of the same frequency with different angles of polarisation will form a third plane polarised wave. We next look at other examples of resultant fields, that are the result of superposition of two orthogonal plane polarised waves, which are not themselves plane polarised.

2. Circular Polarised Light.



In the above example the *y* plane polarised wave has a phase lag of $\delta = -\frac{\pi}{2} = -90^0$ over the *x* plane polarised wave but the same amplitude and frequency/wavelength. Following the resultant field at each of the peaks and troughs leads to a direction of polarisation that rotates. This polarisation just described is known as right hand circular polarisation and is so named as it behaves like a right handed screw, looking back towards the source an observer sees the E field circulating clockwise. If, on the other hand, the *y* component had been leading the *x* component by 90^0 , $\delta = +\frac{\pi}{2}$, the polarisation would have rotated in the counter clockwise sense looking back towards the source, this being left hand circularly polarised light.

We can describe this in mathematical terms by writing the two orthogonal fields as

$$\left.\begin{aligned} E_x &= E_{0x}\cos(kz-\omega t) \\ E_y &= E_{0y}\cos(kz-\omega t+\delta) \end{aligned}\right\} \tag{4.5a}$$

Where in the case we have just discussed, $E_{0x} = E_{0y} = E_0$ and $\delta = -\frac{\pi}{2}$

In this case

$$E_y = E_{0y}\cos\left(kz-\omega t-\frac{\pi}{2}\right) = E_0\sin(kz-\omega t) \tag{4.5b}$$

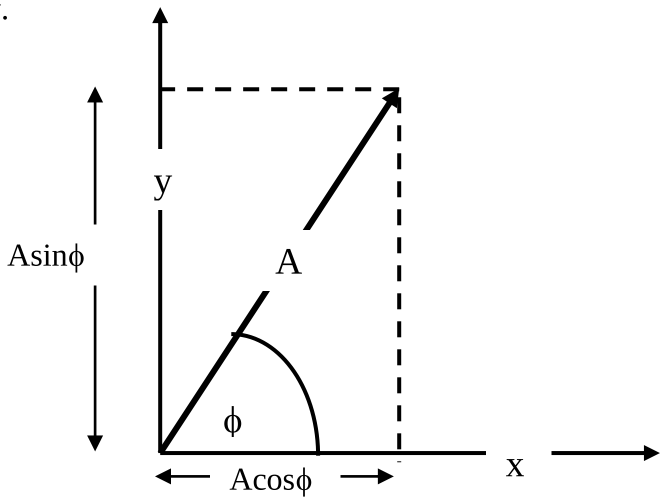
The sum of the two fields is then

$$\vec{E} = E_0\{\hat{x}\cos(kz-\omega t)+\hat{y}\sin(kz-\omega t)\} \tag{4.5c}$$

We can recognise the meaning of 4.5c by considering the vector

$$A\{\hat{x}\cos\phi+\hat{y}\sin\phi\} \tag{4.7}$$

which can be represented in the *xy* plane as a vector of length *A* at an angle ϕ to the x axis as shown below.

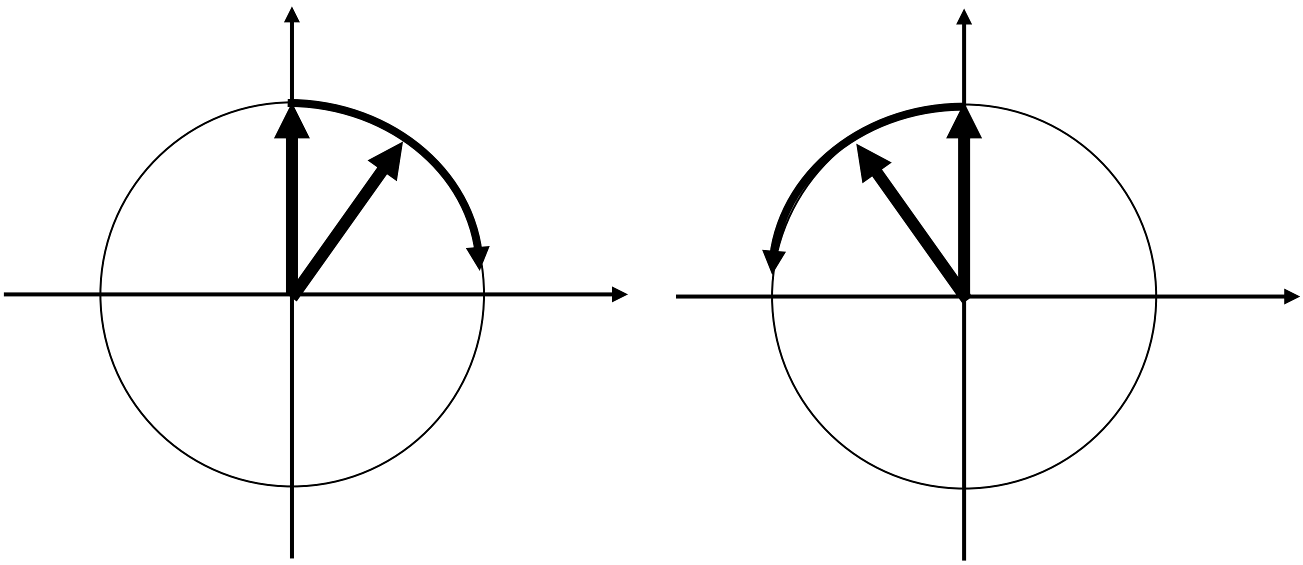


With $\phi = kz - \omega t$ we see by comparing 4.5c with 4.7 that 4.5c the resultant field represents a field with constant amplitude and a direction that rotates in a circular motion at a frequency ω.

If the *y* plane polarised component leads the x component by 90^0 the resultant field is

$$\vec{E} = E_0\{\hat{x}\cos(kz - \omega t) - \hat{y}\sin(kz - \omega t)\} \qquad (4.5d)$$

Which is the left hand circularly polarised wave.



Right hand circular polarisation **Left hand circular polarisation**

The wave is emerging from the paper

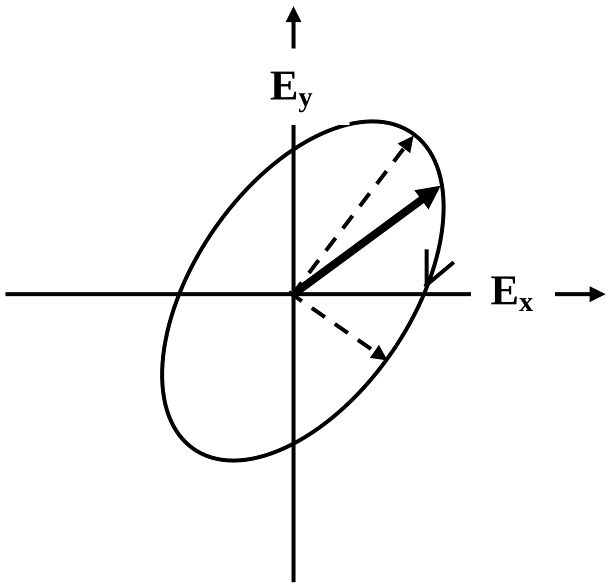
If we take a superposition of a right hand and a left hand circularly polarised wave we have

$$\vec{E} = E_0\{\hat{x}\cos(kz-\omega t)+\hat{y}\sin(kz-\omega t)\}+E_0\{\hat{x}\cos(kz-\omega t)-\hat{y}\sin(kz-\omega t)\}$$

$$\vec{E} = 2E_0\hat{x}\cos(kz-\omega t) \tag{4.8}$$

ie. we have a plane polarised wave once again..

3. Elliptically Polarised Light.



Of course ***the phase difference, δ, can take any value*** and ***the electric field amplitudes of the two orthogonal components may be different***. In this more general case the resultant field will change amplitude and direction with time (or equivalently position) as it propagates. While the field vector traces an ellipse that may or may not be aligned with the x and y axes.

In other words the two orthogonal fields we may write in their most general form

$$\left.\begin{aligned} E_x &= E_{0x}\cos(kz-\omega t) & \text{(4.9a)}\\ E_y &= E_{0y}\cos(kz-\omega t+\delta) & \text{(4.9b)}\end{aligned}\right\}$$

NB. The values of amplitude, E_{0x} and E_{0y} and the phase δ are three constants and we may relate the E_x field and the E_y field to one another at any instant in time and point in space in a straightforward manner as follows;

The curve that the resultant E vector traces should not depend on either position or time. Using the trigonometric identity $\cos(A+B) = \cos A\cos B - \sin A\sin B$

We expand the E_y field

$$E_y = E_{0y}\{\cos(kz-\omega t)\cos\delta - \sin(kz-\omega t)\sin\delta\} \tag{4.10}$$

The dimensionless y component is then;

$$\frac{\mathrm{E_y}}{\mathrm{E_{0y}}} = \{\cos(\mathrm{kz}-\omega\mathrm{t})\cos\delta - \sin(\mathrm{kz}-\omega\mathrm{t})\sin\delta\}$$

(4.10a)

Also the dimensionless x component is;

$$\frac{E_x}{E_{0x}} = \cos(kz-\omega t) \tag{4.11}$$

Using 4.11 in the RHS of 4.10 and re-arranging

$$\frac{E_y}{E_{0y}} - \frac{E_x}{E_{ox}}\cos\delta = -\sin(kz-\omega t)\sin\delta \tag{4.12}$$

Using 4.11 and the trigonometric identity $\sin^2 A + \cos^2 A = 1$

$$\sin(kz-\omega t) = \left[1-\left(\frac{E_x}{E_{0x}}\right)^2\right]^{1/2} \tag{4.13}$$

and using 4.13 in 4.12

$$\left(\frac{E_y}{E_{0y}} - \frac{E_x}{E_{0x}}\cos\delta\right)^2 = \left[1 - \left(\frac{E_x}{E_{0x}}\right)^2\right]\sin^2\delta \tag{4.14}$$

Or by re-arranging terms

$$\left(\frac{E_y}{E_{0y}}\right)^2 + \left(\frac{E_x}{E_{0x}}\right)^2 - 2\left(\frac{E_y}{E_{0y}}\right)\left(\frac{E_x}{E_{0x}}\right)\cos\delta = \sin^2\delta \tag{4.15}$$

The above equation does not involve z or t and is the equation of an ellipse that makes an angle α with the (*x*,*y*) co-ordinate system with;

$$\tan 2\alpha = \frac{2E_{0x}E_{0y}}{E_{0x}^2 - E_{0y}^2}\cos\delta \tag{4.16}$$

i) ***We may show it describes an ellipse***

We can more easily see that 4.15 is the equation of an ellipse if the phase shift $\delta = 90^0$ and $\cos\delta = 0$. This removes the cross product term and $\sin^2\delta = 1$

$$\left(\frac{E_y}{E_{0y}}\right)^2 + \left(\frac{E_x}{E_{0x}}\right)^2 = 1 \qquad (4.17$$

4.17 is easily recognisable as an ellipse with axes, E_{0x}, E_{0y}

ii) ***We may show that for $\delta = \pi/2$ and equal amplitudes that it describes a circle***

If the amplitudes of the two orthogonal plane polarised waves were equal, $E_{0x} = E_{0y} = E_0$ then 4.17 becomes;

$$E_x^2 + E_y^2 = E_0^2 \tag{4.18}$$

This is the equation of a circle as previously encountered in the description of the circularly polarised light where $\delta = 90^0$ and $E_{0x} = E_{0y} = E_0$.

iii) ***We may show that for δ = 0 and equal amplitudes that it describes linear plane polarisation***

The above analysis means that if $\delta = 0$ in 4.15 we should recover a linear polarised wave. In this case cos 0 = 1 and sin 0 = 0 and inserting these values into 4.15

$$\left(\frac{E_y}{E_{0y}}\right)^2 + \left(\frac{E_x}{E_{0x}}\right)^2 - 2\left(\frac{E_y}{E_{0y}}\right)\left(\frac{E_x}{E_{0x}}\right) = 0 \tag{4.19}$$

Or rewriting as a square and taking the square root

$$\left(\frac{E_y}{E_{0y}} - \frac{E_x}{E_{0x}}\right)^2 = 0 \tag{4.20}$$

$$\frac{E_y}{E_{0y}} = \frac{E_x}{E_{0x}} \qquad \text{or} \qquad \frac{E_y}{E_x} = \frac{E_{0y}}{E_{0x}} \tag{4.21}$$

The ratio of y component to x component is a constant independent of t or z and 4.21 thus represents a linear plane polarisation with the angle of the electric field, θ with respect to the x axis given by

$$\tan\theta = \frac{E_{oy}}{E_{0x}} \tag{4.22}$$

Thus the elliptical polarisation state is the most general polarisation state with plane polarised and circularly polarised light being special examples of elliptically polarised light.

Birefringence

In the earlier parts of the course we have considered the propagation of light in simple media after a consideration of propagation in vacuum and saw how the description needed modification. The modification required the introduction of optical properties for the medium namely; χ_E , ε and n, electronic susceptibility, dielectric constant and refractive index respectively. We chose simple media, that is isotropic and homogeneous media and we found relationships amongst these material properties;

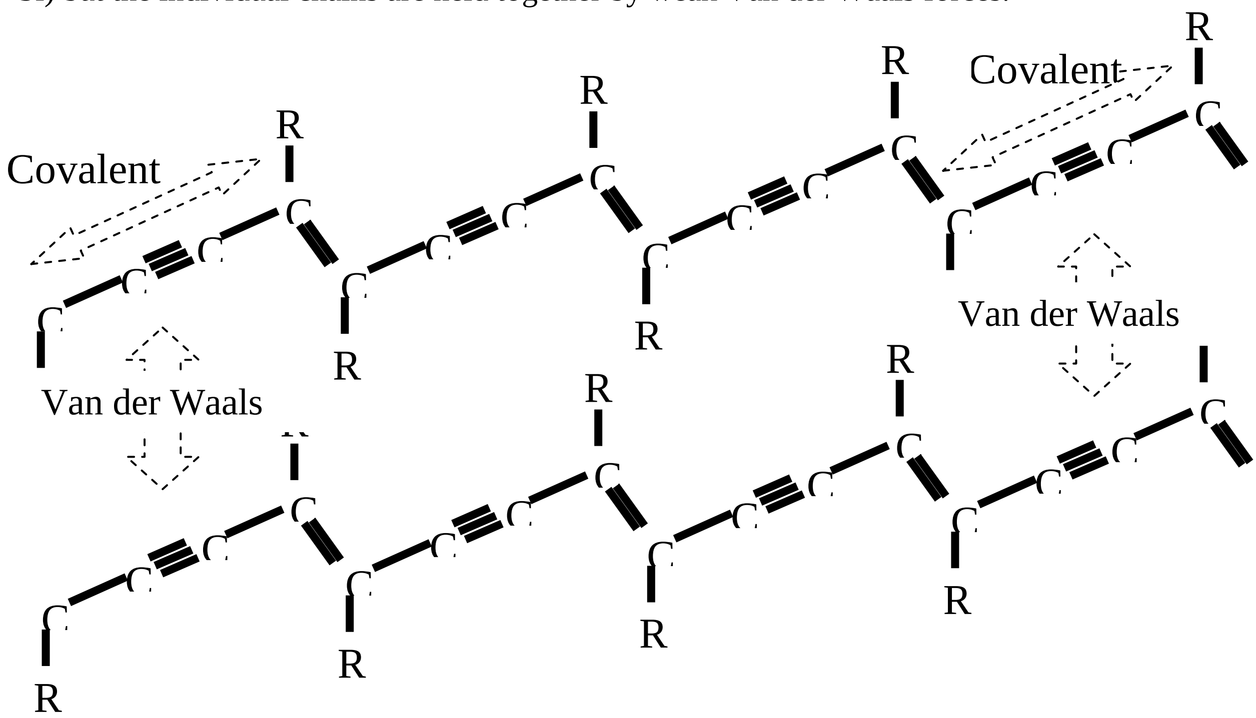
$$\left.\begin{aligned}
&\vec{P} = \varepsilon_0 \chi_E \vec{E} \\
&\vec{D} = \varepsilon\varepsilon_0 \vec{E} = \varepsilon_0 \vec{E} + \vec{P} \\
&\varepsilon = 1 + \chi_E \\
&n = \sqrt{\varepsilon\mu} = \sqrt{\varepsilon} = \sqrt{1+\chi}
\end{aligned}\right\} \qquad (4.23)$$

All of the above relationships apply for simple media but many materials are anisotropic, particularly crystalline materials where there is some ordering amongst the constituent atoms and the possibility arises that there is some anisotropy in the material properties, ie. their value depends on the orientation of the electric field associated with the light wave.

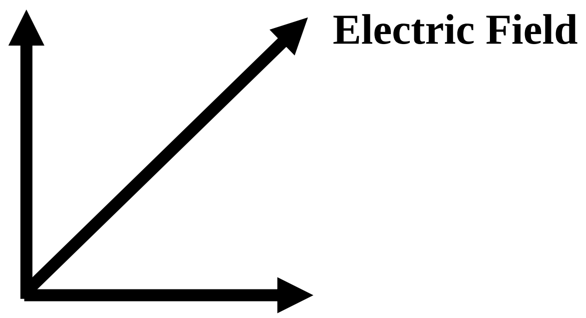
To be able to observe the effects of polarisation as described to their fullest extent we need the electromagnetic field to be propagating in these more complex media where there may be for example anisotropy in the optical properties that were not encountered earlier.

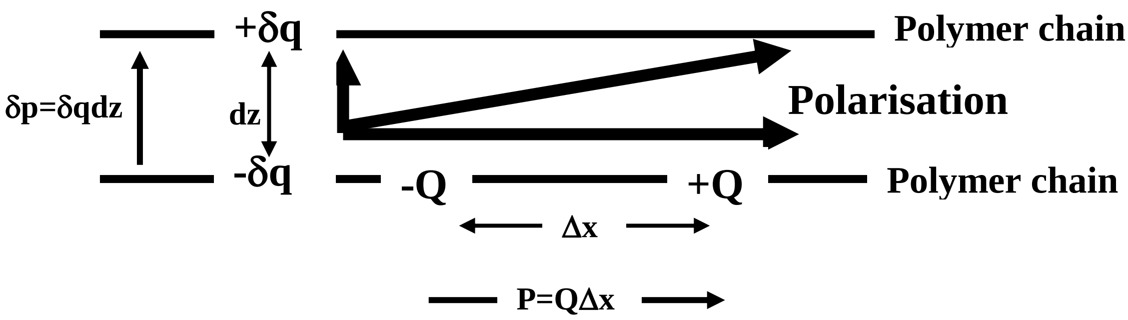
So far, in all that has preceded, it has been assumed that the refractive index of a medium is one scalar constant that is independent of the direction of polarisation of the electromagnetic wave whose velocity it modifies. In actuality n will be dependent on the direction of polarisation in most classes of solid except those that are amorphous or possess cubic symmetry. This property of solids is known as birefringence. A simple way of thinking about birefringence is to appreciate that the polarisation $\boldsymbol{P}$ induced by an electric field $\boldsymbol{E}$ (and consequently $\boldsymbol{D} = (\varepsilon_0\boldsymbol{E} + \boldsymbol{P}$) will not necessarily be in the same direction as **E**. That is to say that we need to modify our view of the electronic susceptibility, a measure of the ease with which a material is polarised (ie charge

seperated). Previously, in the above equations, 4.23, it is considered a scalar constant and the polarisation per unit volume $\underline{\boldsymbol{P}}$ is in the same direction as the electric field $\underline{\boldsymbol{E}}$.
More realistically we must treat χ_E (and consequently, ε and n) as a tensor reflecting the varying ease of polarisability (charge separation) in different directions in a material. This is most clearly demonstrated by an example where the effect is large.
Polydiacetylene is a conjugated polymer which may be obtained as macroscopic single crystals where the polymer chains are all aligned. The chain is covalently bonded (as is Si) but the individual chains are held together by weak Van der Waals forces.



Charge is relatively easily seperated by an electric field applied along the polymer chain direction but far less readily seperated by a field perpendicular to the chain as electrons are strongly restricted to the chain they find themselves on and cannot move from one chain to another due to the interchain separation whereas they can readily move along the covalently bonded polymer chain and it is the charge displacement that gives rise to the polarisation. A little thought will show that a field, ***E***. polarised at 45^0 to the chains will have a large component of polarisation in the chain direction, due to the component of *E* in that direction, and virtually zero polarisation perpendicular. ***P*** is therefore almost entirely along the chain whilst ***E*** is at 45^0 to the chain. The situation is depicted schematically below.





The above gives a physical example and description of the origin of birefringence. There is a small ammount of charge displacement, $\pm\delta q$ from one chain to another and a much larger charge displacement $\pm Q$ on the chain as it is here that the charge is mobile. This results in a very small interchain polarisation δp and a large intrachain polarisation P.

More formally the relation between polarisation and applied field must be written in tensor form :

$$\left.\begin{aligned} P_x &= \varepsilon_0\left(\chi_{xx}E_x + \chi_{xy}E_y + \chi_{xz}E_z\right) \\ P_y &= \varepsilon_0\left(\chi_{yx}E_x + \chi_{yy}E_y + \chi_{yz}E_z\right) \\ P_z &= \varepsilon_0\left(\chi_{zx}E_x + \chi_{zy}E_y + \chi_{zz}E_z\right) \end{aligned}\right\} \qquad (4.24)$$

where the χ_{ij} are the components of the susceptibility tensor.

There exists a set of axes in any crystal called "the principle dielectric axes", x, y, z for which all but the diagonal components, χ_{ii} , are zero. However for all but amorphous materials such as glasses and certain of those with cubic symmetry, the three remaining components χ_{xx} , χ_{yy} , and χ_{zz} are not equal. This implies, as we have seen that ***P***, ***E***, and ***D*** are not in the same direction. It also has implications for propagation of an electromagnetic wave in such a medium.

We noted earlier that there is a relationship between χ_E and n,

$$n = \sqrt{1+\chi_E} \qquad (4.25)$$

In other words, for these anisotropic media the refractive index must also be represented by a tensor. If we have chosen the cartesian axes, x, y, z to be in the same direction as the principle dielectric axes there will be three principle values of refractive, n_x , n_y , and n_z all of which may be different. Such behaviour is called birefringence and the implication is that light propagating within a birefringent medium will propagate at a ***velocity that depends on the direction of polarisation***. For example plane polarised light, polarised in the x direction will have a phase velocity $v = \frac{c}{n_x}$ etc.

Recalling from earlier discussion that any polarisation state (circular, elliptical, plane) can be described as the sum of two orthogonal plane polarised states, we can now see the significance of that statement! We now discuss birefringence in more detail and see that it is necessary to consider any light wave to be split into orthogonal plane polarised components in order to describe its propagation in a birefringent medium.

In general birefringent materials may be subdivided into two classes as follows.

(i) **Uniaxial materials** where only one index, n_z say, is different and $n_x = n_y$ In this case $n_x = n_y = n_o$ the ordinary refractive index and $n_z = n_e$ the extraordinary refractive index.

Such materials may be further subdivided into;

positive uniaxial materials where $n_o < n_e$, and the ordinary ray propagates at a higher velocity as a result

and

negative uniaxial materials where $n_o > n_e$ and it is the extraordinary ray that propagates at the higher velocity.

Two classic uniaxial birefringent materials are the minerals **quartz** where n_o = 1.5443 and n_e = 1.5534 with a birefringence value $\Delta n = n_e - n_o = +0.0910$ and **calcite** which is a negative uniaxial crystal with n_o = 1.6584 and n_e = 1.4864 and its birefringence $\Delta n = n_e - n_o = -0.1720$.

The z direction in the above example ($n_x = n_y = n_o$ and $n_z = n_e \neq n_o$) is called the optic axis and is important because any wave travelling along the optic axis can only have field components which see an identical refractive index, $n_X = n_Y$. It will then travel through the medium unchanged as far as its polarisation state is concerned.

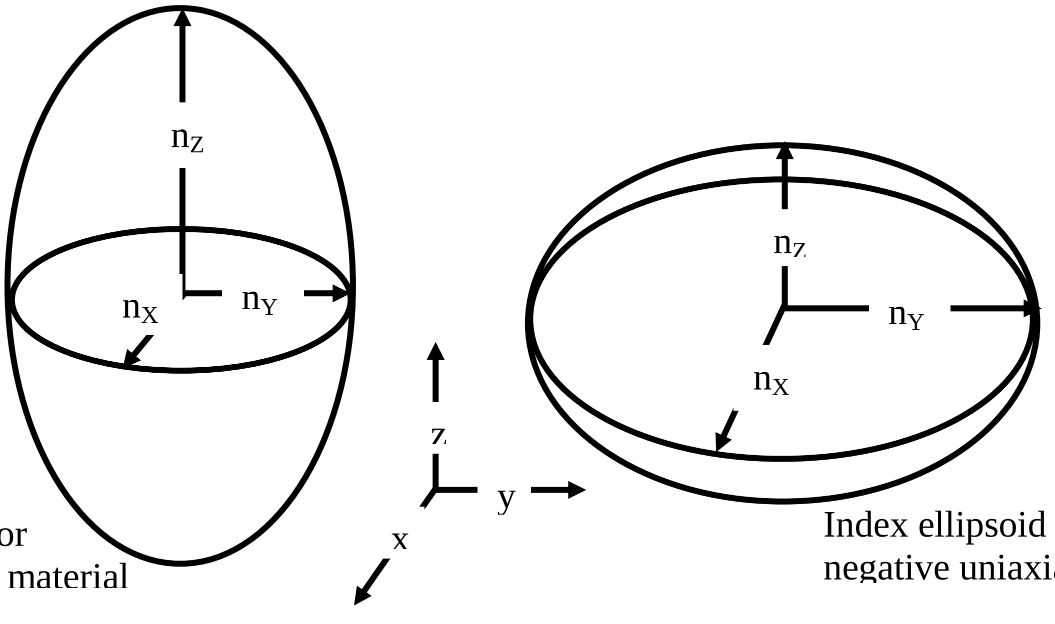
(ii) **Biaxial materials** where $n_X \neq n_Y \neq n_Z$. These will have two optic axes which are not the principle dielectric axes. They are a more difficult case to consider and are less frequent.

We will be satisfied to limit ourselves to limit ourselves to the analysis of uniaxial crystals in what follows.

Uniaxial birefringence.

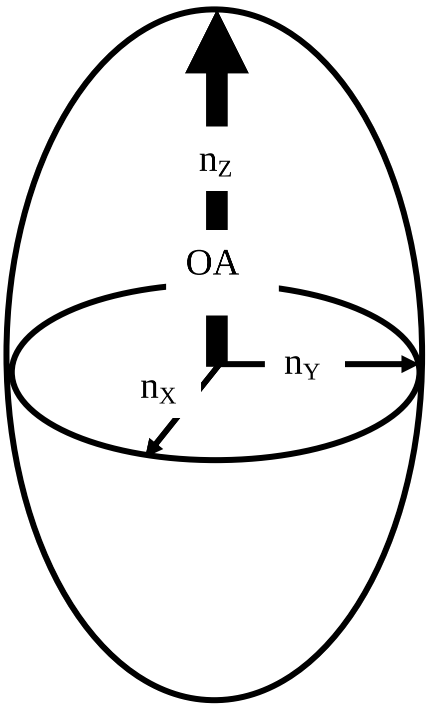
Uniaxial materials have two defining refractive indices, $n_x = n_y$ and n_z known as the ordinary refractive index, n_o and the extraordinary refractive index, n_e respectively.Any wave not propagating in the direction of the optic axis, ie. The z direction in this example, will split into two mutually orthogonal plane polarised waves with differing phase velocities known as the ordinary and the extraordinary waves. For any arbitrary propagation direction a construction called the index ellipsoid may be used in order to discover what happens. This is an ellipsoid constructed such that its semi-major/minor axes have lengths n_x , n_y and n_z and they are drawn below for positive and negative uniaxial materials.

The electric field of a wave travelling in an arbitrary direction may for simplicity of analysis be considered as split into two plane polarised components; an ordinary wave polarised perpendicular to the optic axis, z, and an extraordinary wave polarised orthogonal to the ordinary wave. In a positive material the ordinary wave moves at a faster phase velocity than the extraordinary wave and vice versa for a negative material.

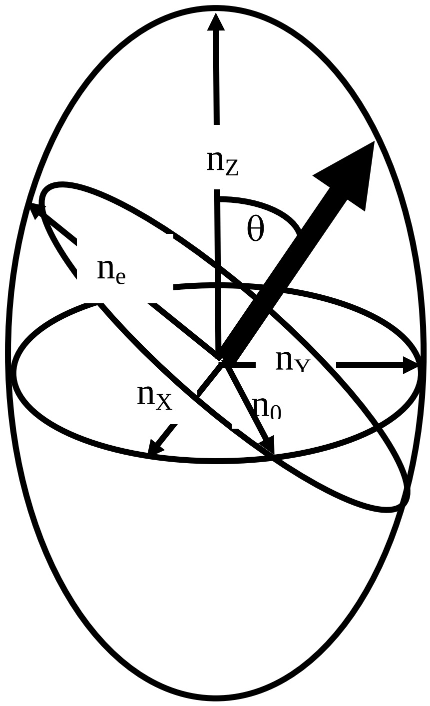


Index ellipsoid for positive uniaxial material

Index ellipsoid for negative uniaxial material

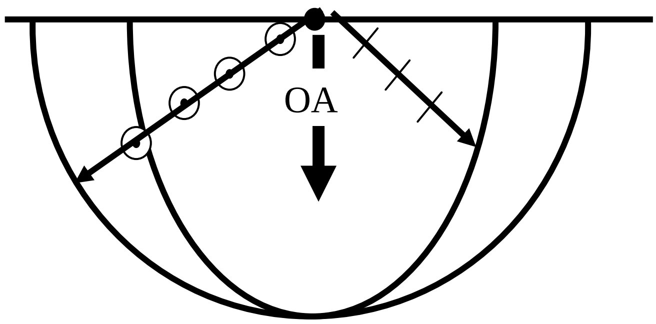


Propagation along the optic axis

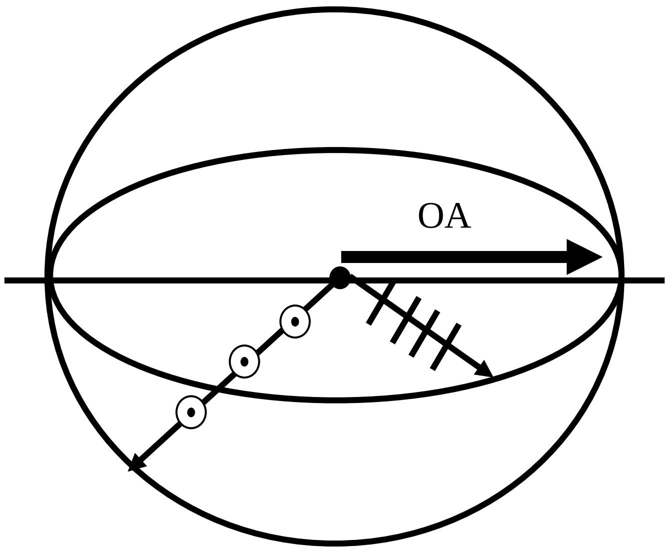


Propagation off axis

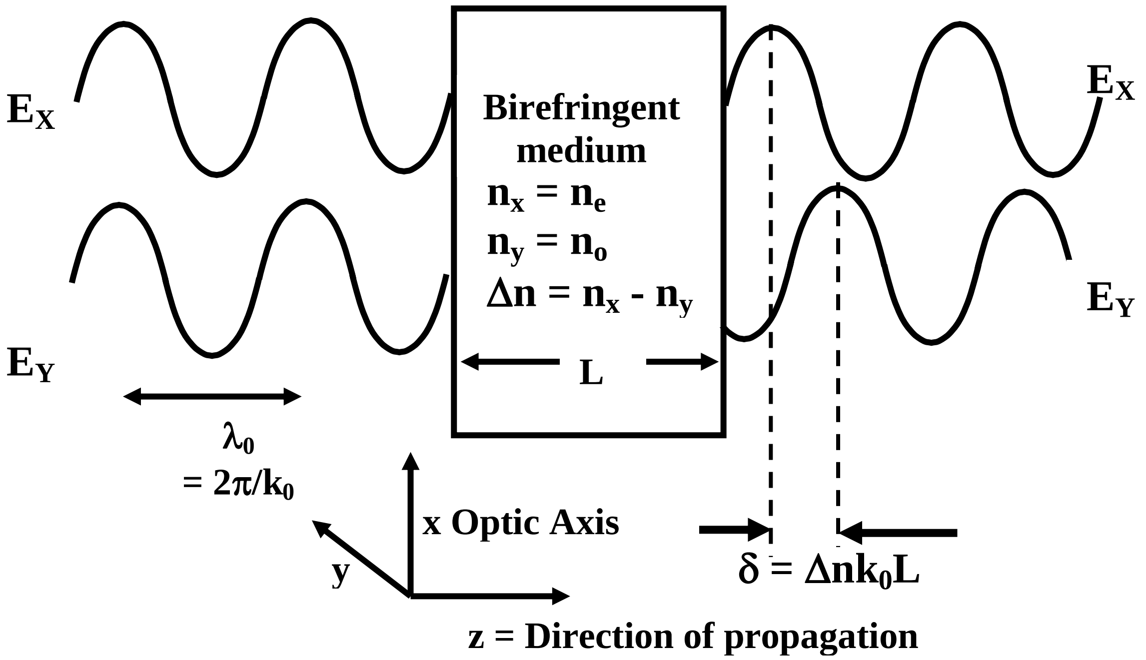
Consider first propagation along the optic axis of a positive uniaxial crystal as represented in the diagram. The large arrowed vector, OA, in the diagram indicates the direction of propagation and the polarisation components see a refractive index discovered by constructing a plane normal to that direction (the x-y plane) containing the origin. This makes a circular section through the ellipsoid whose radius is defined by n_0 . Because that section of the ellipsoid is a circle any two orthogonal components will see a refractive index $n_x = n_y = n_o$ and will travel with no change in phase. Such a situation, ie. propagation along the optic axis, gives rise to no unusual effects. Ie. our two orthogonal plane polarised components (that form any polarisation state) will retain any phase relationship they had possessed before entering the medium and the polarisation state will be left unaltered. Now consider the off axis propagation at an angle θ to the optic axis as represented in the second diagram. The bold arrowed vector indicates the direction of propagation. To use the ellipsoid, a plane normal to this direction containing the origin is constructed. It intersects the ellipsoid to form an ellipse whose minor axis is n_0 and whose major axis is $n_e(\theta)$, a function of θ. These two refractive indices will be the refractive indices of the ordinary and extraordinary wave into which the original wave will split. The two orthogonal plane polarised components that describe the initial polarisation state will become those resolved along and perpendicular to the axes of the ellipse. In other words a point source radiating into the medium will split into an ordinary wave whose velocity is independent of direction and therefore spreads out as a circular wavefront, and an extraordinary wave whose velocity depends on θ as shown in the diagrams below. The ordinary and extraordinary wavefronts touch in the direction of the optic axis as shown. NB the O ray is polarised perpendicular to the optic axis.



Propagation from a point source into a positive uniaxial crystal. The optic axis, OA, is perpendicular to the surface.



Propagation from a point source into a positive uniaxial crystal. The optic axis, OA, is parallel to the surface.



Uniaxial birefringent materials may be useful in optical instruments as a means of altering the polarisation state of the light. One example of this will suffice to demonstrate the potential usefulness.

The above diagram shows two orthogonal plane polarised electromagnetic waves incident on a birefringent medium from the left. The optic axis is in the x direction and therefore $n_x = n_e$ and therefore the orthogonal polarisation $n_y = n_o$. Before entering the medium the two components are in phase and the resultant field will also be plane polarised at an angle of 45^0 to the *x* and *y* axes as found in an earlier analysis. Once entered within the crystal however they each propagate at a different velocity and when emerging from the crystal after a path length *L* they will no longer be in phase. To understand this it is important to be clear about what the phase is. Recall that for a plane wave described by

$$E = E_0 \cos(kz - \omega t) \tag{4.26}$$

The phase, ϕ, is

$$\phi = (kz - \omega t) \tag{4.27}$$

In free space, light travels at a velocity c and there is a relation between λ and ν

$$\lambda\nu = c \qquad (4.28a)$$

Once within the medium this velocity is altered as described by the refractive index and the relationship is now

$$\lambda\nu = \frac{c}{n} \qquad (4.28b)$$

<u>The frequency on the LHS has not changed but the velocity on the RHS has become smaller. In fact the wavelength, λ , is no longer the same.</u> Writing the wavelength in free space as λ_0 once the plane wave is propagating in the medium the wavelength becomes

$$\lambda = \frac{\lambda_0}{n} \qquad (4.28c)$$

Recalling the definition of the magnitude of the wavevector, $k = \frac{2\pi}{\lambda}$, the wavevector in the medium is also altered when compared to the wavevector in free space, k_0 .

$$k = nk_0 \qquad (4.29)$$

In the birefringent medium we have different wavevectors for the x polarised wave and the y polarised wave and also different phases as a result

$$k_x = n_x k_0 \qquad\qquad k_y = n_y k_0 \qquad (4.30)$$

The phases are then

$$\phi_x = n_x k_0 z - \omega t \qquad\qquad \phi_y = n_y k_0 z - \omega t \qquad (4.31)$$

The phase difference is then

$$\delta = \phi_x - \phi_y = (n_x - n_y)k_0 z = \Delta n k_0 z \qquad (4.32)$$

Thus on emerging from the crystal of length L the phase difference is

$$\delta = \Delta n k_0 L \tag{4.33}$$

As we have seen in an earlier section the light wave will in general be elliptically polarised when the two orthogonal plane polarised waves have a phase difference δ as described by 4.15.

There are interesting possibilities as we noted earlier. If the two orthogonal polarisations have a phase difference of $\delta = (2m+1)\pi$ where m = 0, ±1, ±2…. the plane of polarisation of the resultant will be rotated by 90^0 compared to where they are in phase. They are in phase at the input in the above example and therefore if $\delta = (2m+1)\pi = \Delta n k_0 L$ the plane of polarisation will be rotated through 90^0. For a given wavelength (or wavevector) and a material with a given birefringence, Δn, the length of the crystal can be adjusted to satisfy the condition, ie.

$$L_{\lambda/2} = \frac{(2m+1)\pi}{\Delta n k_0} = \frac{(2m+1)\lambda_0}{2\Delta n} \tag{4.34}$$

Such an arrangement is known as a half wave plate. It is important to note that this will only be a half wave plate for certain wavelengths that satisfy 4.34 and that the birefringent medium is able to transmit.

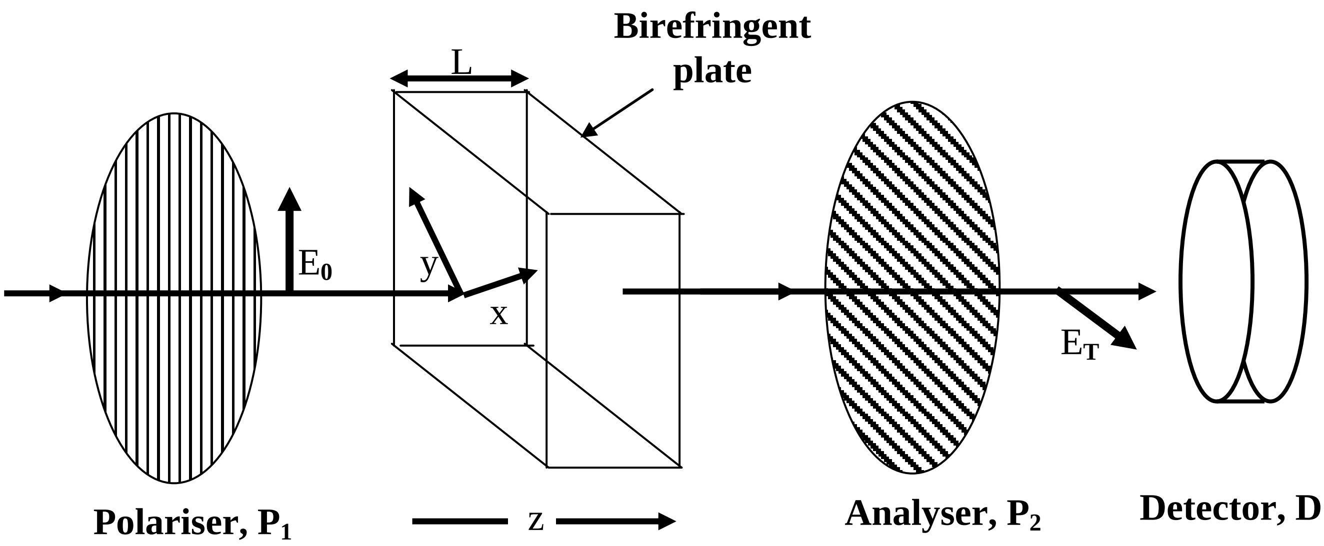
Altering the thickness of the wave plate may introduce a phase difference $\delta = (2m+1)\frac{\pi}{2}$ where m = 0, ±1, ±2 and this will cause a plane polarised input to become circularly polarised (and vice verca) when L is such that

$$L_{\lambda/4} = \frac{(2m+1)\pi/2}{\Delta n k_0} = \frac{(2m+1)\lambda}{4\Delta n} \tag{4.35}$$

This is called a quarter wave plate. Again it is only a quarter wave plate for the set of wavelengths which satisfy 4.35 and for which the medium is transparent.

Transmission of polarised light through polariser

It is useful to be able to calculate the amount of light in a given polarisation state that will be transmitted by a second polarising element (the analyser) set to transmit plane polarised light polarised in a particular sense wrt the plane of polarisation of the input light.



We approach this problem by considering the particular arrangement shown above. This arrangement consists of ;

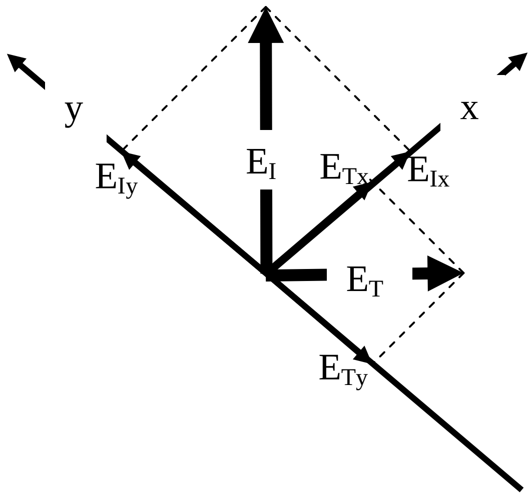
i) A polarising element P_1 that prepares light in a state of plane polarisation with the plane of polarisation in a particular direction.

ii) A plate of birefringent material of thickness L will alter the polarisation state into a new polarisation state, after the light has traversed the plate, by introducing a phase lag, δ, between the two orthogonal plane polarised waves that constitute the original wave.

iii) An analyser, P_2 , is a second polarising element that will only pass light polarised in a plane orthogonal to that of the original polariser P_1.

We wish to know the fraction of the light intensity incident on the polariser P_1 that is detected after it has traversed the whole system including P_2.

With no birefringent plate present the plane polarised light from P_1 will be unchanged as far as its polarisaton state is concerned and hence no light will be transmitted by P_2 which is set orthogonal to the polariser, P_1. The introduction of the birefringent plate will introduce a phase difference between the two orthogonal components that make up the original plane polarised wave, these two components being the ordinary and extraordinary waves plane polarised in the y and x directions respectively.



The above diagram is invaluable in enabling us to visualise the components of the input and output electric fields, E_I and E_T in the x and y directions. With this pictorial representation the problem is straightforward to solve.
Examination of the reference frame as shown above shows that to find E_T after passage through the analyser requires addition of $+E_{Tx}$ and $-E_{Ty}$. (in contrast to the E_I which requires $+E_{Ix}$ and $+E_{Iy}$). Thus
At the entry into the plate the plane waves for each component are

$$\left.\begin{aligned} E_{Ix} &= \frac{E_I}{\sqrt{2}}\cos(k_0 z - \omega t) \\ \\ E_{Iy} &= \frac{E_I}{\sqrt{2}}\cos(k_0 z - \omega t) \end{aligned}\right\}$$

As the input wave is plane polarised with an amplitude E_I at 45^0 to both x and y axes there are equal amplitudes for x and y components, $\dfrac{E_I}{\sqrt{2}}$.

If we arbitrarily choose z = 0 at the input to the birefringent crystal.

Upon exiting the analyser, P_2, the fields of each component are given by

$$\left.\begin{aligned} E_{Tx} &= \frac{E_{Ix}}{\sqrt{2}}\cos(n_e k_0 L - \omega t) = \frac{E_I}{2}\cos\left(n_e k_0 L - \omega t\right) \\ E_{Ty} &= -\frac{E_{Iy}}{\sqrt{2}} cos(\, n_o k_0 L - \omega t\,) = -\frac{E_I}{2} cos\left(n_o k_0 L - \omega t\right) \end{aligned}\right\}$$

Where each component has suffered a phase change $\phi_x = n_e k_0 L$ and $\phi_y = n_0 k_0 L$. The resultant transmitted field is thus

$$E_T = \frac{E_I}{2}\left[cos\left(n_e k_0 L - \omega t\right) - cos\left(n_o k_0 L - \omega t\right)\right]$$

NB. The x axis has been chosen to be the optic axis and light polarised in the x direction "sees" the extraordinary refractive index.

Ie. after traversing the plate there is a phase difference given by

$$\delta = \phi_x - \phi_y = \Delta n k_0 L = (n_e - n_o)\frac{2\pi}{\lambda_0}L$$

Using the trigonometric identity

$$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$E_T = \frac{E_I}{2} 2\sin\left(\frac{n_e + n_o}{2}k_0 L - \frac{\omega t}{2}\right)\sin\left(\frac{n_e - n_o}{2}k_0 L\right)$$

Which we can rewrite more compactly using our expression for δ as

$$E_T = E_I \sin\left(\frac{n_e + n_o}{2}k_0 L - \frac{\omega t}{2}\right)\sin\frac{\delta}{2}$$

We find the output intensity I_T by finding the time average of the square of the electric field divided by impedance of free space, η_0 giving;

$$I_T = \frac{\langle E_T^2 \rangle}{\eta_0} = \frac{E_T^2}{2\eta_0} = \frac{E_I^2}{\eta_0}\sin^2\left(\frac{\delta}{2}\right)\left\langle \sin^2\left(\frac{n_e + n_o}{2}k_0 L - \frac{\omega t}{2}\right)\right\rangle = I_i \sin^2\left(\frac{\delta}{2}\right)$$

NB. only one of the sinusoids (with the ωt in its argument) in the expression for E_T has any time dependence and has time averaged to $\frac{1}{2}$ ***for the square of a sinusoid over many cycles*** as usual leaving the simple final expression. and the transmission of the system is then

$$T = \frac{I_0}{I_i} = \sin^2\left(\frac{\delta}{2}\right)$$

Dichroism.

a) Linear Dichroism

After the recent discussion of anisotropic media and how this leads to birefringence, the possession of two refractive indices that describe propagation in the anisotropic medium for orthogonal plane polarised fields, it is possible to extend the description of anisotropy to describe materials where one plane polarisation state may be more strongly absorbed than its orthogonal partner. This is the origin of dichroism and the way that Polaroid plastic works. In the preceding discussion of birefringence we were interested in the light propagating in the medium where refraction is the dominant effect of the anisotropic medium away from frequencies that are strongly absorbed by the medium. However another interesting phenomenon of light propagating in a medium (solid, liquid or gas) is that at certain frequencies the light may be absorbed and the intensity fall as the light propagates through the medium.

Generally, in a simple medium, as light propagates through it at frequencies that may be absorbed the intensity lost (amount of light energy absorbed) is proportional to the distance travelled, and the intensity available to be lost, ie. After travelling an infinitesimal distance dz the drop in intensity is proportional to dz and to the amount of intensity originally present and is thus given by

$$\delta I = -\alpha I dz$$

where α is the constant of proportionality. We may re-arrange this equation to give

$$\frac{dI}{I} = -\alpha dz$$

Integrating from $z = 0$ to L

$$\log_e I\Big|_{I_0}^{I(z)} = -\alpha z\Big|_0^L$$

Evaluating, taking exponentials and rearranging

$$I = I_0 \exp(-\alpha L)$$

This is known as Beers Law and α is the absorption coefficient.

It is simple now to see that for anisotropic media the absorption coefficient will depend on the direction in which the electric field of the light wave is pointing, ie. on the plane of polarisation. The act of absorption, while a quantum event, can be thought of as the electrons in the medium moving in response to the electric field. They will find it easier to move in one direction rather than another, eg. reverting to our earlier example, along a polymer chain rather than perpendicular to it. This leads to preferential absorption for different plane polarisation states and the necessity to identify different absorption coefficients for different polarisations. This is known as linear Dichroism and is the basis for the most common polariser, the polaroid sheet, in which the material has been subject to stretching forces allowing the polymer chains that compose the plastic to align in a preferred direction ie the direction of the stress. Light that is plane polarised in this direction will be strongly absorbed whilst light polarised orthogonal to this alignment direction will be weakly absorbed , we thus define $\alpha_{//}$ and $\alpha_{\perp}$ light plane polarised parallel to the direction of aligmment and perpendicular to the direction of alignment and $\alpha_{//} >> \alpha_{\perp}$.

b) Circular Dichroism

It can also be the case that the left and right hand circularly polarised waves are absorbed to a different extent providing the molecule/substance doing the absorbing is able to exist in right handed and left handed forms known as enantiomers. Such molecules are inevitably possessing of chirality and their structures allow a left handed and a right handed form of the molecule to be identified, This is common among biological molecules such as sugars and amino acids (and the proteins formed of amino acids). If a plane polarised wave enters a sugar solution, provided it is a pure enantiomer and made up of only one “handed” sugar, the two oppositely rotating circular polarisations that we have already seen can be thought of as composing the linearly polarised beam will be absorbed to differing extents and the right hand polarised beam being the more weakly absorbed will come to dominate the light wave changing from a plane polarised state to an elliptically polarised state. The polarisation is said to be rotated clockwise and the solution of sugar is dextrorotary (from the Latin for right) and if rotated counter clockwise the solution is laevo-rotatry. Hence the other name for glucose, dextrose). We have seen how this occurs where the amplitudes of the two plane polarisations are different in for example 4.15.

In Summary the most important things we need to know about polarisation states are;

i) There are three polarisation states of importance

a) Plane polarised light where the plane of polarisation is independent of position and time i.e. the electric field vector always points in the same direction as the wave propagates

b) Circularly polarised light, where the plane of polarisation rotates about the axis of propagation at an angular frequency, ω, the same as the frequency of the light wave but the amplitude is fixed independent of time and position. We may speak of left and right circularly polarised light.

c) Elliptically polarised light is the most general polarisation state and the others are limiting forms of the elliptically polarised state where both amplitude and direction may vary with time and position as the wave propagates.

ii) All polarisation states may be described as composed of two orthogonal plane polarised waves.

iii) One can change between polarisation states by changing the phase relationship between the orthogonal plane polarised states.

iv) Birefringent crystals/materials will naturally separate the two orthogonal polarisations of an arbitrary polarisation states into an ordinary and an extraordinary plane polarised wave each of which travels with a different velocity.

v) The ordinary refractive index, n_0 , of a birefringent crystal is independent of the angle at which the wave propagates through the crystal wrt the optic axis whereas the extraordinary refractive index will depend on the angle $n_e(\theta)$